Homework 3
Solutions
EE-312

1.18. (a) Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$x_1[n] \longrightarrow y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k]$$

$$x_2[n] \longrightarrow y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k]$$

Let $x_3[n]$ be a linear combination of $x_1[n]$ and $x_2[n]$. That is,

$$x_3[n] = ax_1[n] + bx_2[n]$$

where a and b are arbitrary scalars. If $x_3[n]$ is the input to the given system, then the corresponding output $y_3[n]$ is

$$\begin{aligned} y_3[n] &= \sum_{k=n-n_0}^{n+n_0} x_3[k] \\ &= \sum_{k=n-n_0}^{n+n_0} (ax_1[k] + bx_2[k]) = a \sum_{k=n-n_0}^{n+n_0} x_1[k] + b \sum_{k=n-n_0}^{n+n_0} x_2[k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Therefore, the system is linear.

(b) Consider an arbitrary input $x_1[n]$. Let

$$y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k]$$

be the corresponding output. Consider a second input $x_2[n]$ obtained by shifting $x_1[n]$ in time:

$$x_2[n] = x_1[n - n_1]$$

The output corresponding to this input is

$$y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k] = \sum_{k=n-n_0}^{n+n_0} x_1[k - n_1] = \sum_{k=n-n_1-n_0}^{n-n_1+n_0} x_1[k]$$

Also note that

$$y_1[n - n_1] = \sum_{k=n-n_1-n_0}^{n-n_1+n_0} x_1[k].$$

Therefore,

$$y_2[n] = y_1[n - n_1]$$

This implies that the system is time-invariant.

(c) If $|x[n]| < B$, then

$$y[n] \leq (2n_0 + 1)B$$

Therefore, $C \leq (2n_0 + 1)B$.

Pb. 1.19

(b) (i) Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$x_1[n] \longrightarrow y_1[n] = x_1^2[n-2]$$
$$x_2[n] \longrightarrow y_2[n] = x_2^2[n-2]$$

Let $x_3[n]$ be a linear combination of $x_1[n]$ and $x_2[n]$. That is,

$$x_3[n] = ax_1[n] + bx_2[n]$$

where a and b are arbitrary scalars. If $x_3[n]$ is the input to the given system, then the corresponding output $y_3[n]$ is

$$\begin{aligned} y_3[n] &= x_3^2[n-2] \\ &= (ax_1[n-2] + bx_2[n-2])^2 \\ &= a^2x_1^2[n-2] + b^2x_2^2[n-2] + 2abx_1[n-2]x_2[n-2] \\ &\neq ay_1[n] + by_2[n] \end{aligned}$$

Therefore, the system is **not linear.**

(ii) **Consider an arbitrary input $x_1[n]$. Let**

$$y_1[n] = x_1^2[n-2]$$

be the corresponding output. Consider a second input $x_2[n]$ obtained by shifting $x_1[n]$ in time:

$$x_2[n] = x_1[n-n_0]$$

The output corresponding to this input is

$$y_2[n] = x_2^2[n-2] = x_1^2[n-2-n_0]$$

Also note that

$$y_1[n-n_0] = x_1^2[n-2-n_0]$$

Therefore,

$$y_2[n] = y_1[n-n_0]$$

This implies that the system is **time-invariant.**

(c) (i) Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$x_1[n] \longrightarrow y_1[n] = x_1[n+1] - x_1[n-1]$$
$$x_2[n] \longrightarrow y_2[n] = x_2[n+1] - x_2[n-1]$$

Let $x_3[n]$ be a linear combination of $x_1[n]$ and $x_2[n]$. That is,

$$x_3[n] = ax_1[n] + bx_2[n]$$

where a and b are arbitrary scalars. If $x_3[n]$ is the input to the given system, then the corresponding output $y_3[n]$ is

$$\begin{aligned} y_3[n] &= x_3[n+1] - x_3[n-1] \\ &= ax_1[n+1] + bx_1[n+1] - ax_1[n-1] - bx_2[n-1] \\ &= a(x_1[n+1] - x_1[n-1]) + b(x_2[n+1] - x_2[n-1]) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Therefore, the system is **linear.**

(ii) Consider an arbitrary input $x_1[n]$. Let

$$y_1[n] = x_1[n+1] - x_1[n-1]$$

be the corresponding output. Consider a second input $x_2[n]$ obtained by shifting $x_1[n]$ in time:

$$x_2[n] = x_1[n-n_0]$$

The output corresponding to this input is

$$y_2[n] = x_2[n+1] - x_2[n-1] = x_1[n+1-n_0] - x_1[n-1-n_0]$$

Also note that

$$y_1[n - n_0] = x_1[n + 1 - n_0] - x_1[n - 1 - n_0]$$

Therefore,

$$y_2[n] = y_1[n - n_0]$$

This implies that the system is **time-invariant**.

1.27. (a) Linear, stable.

(b) Memoryless, linear, causal, stable.

(c) Linear

(d) Linear, causal, stable.

(e) Time invariant, linear, causal, stable.

(f) Linear, stable.

(g) Time invariant, linear, causal.

1.28. (a) Linear, stable.

(b) Time invariant, linear, causal, stable.

(c) Memoryless, linear, causal.

(d) Linear, stable.

(e) Linear, stable.

(f) Memoryless, linear, causal, stable.

(g) Linear, stable.

2.1. (a) We know that

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad \text{(S2.1-1)}$$

The signals $x[n]$ and $h[n]$ are as shown in Figure S2.1.

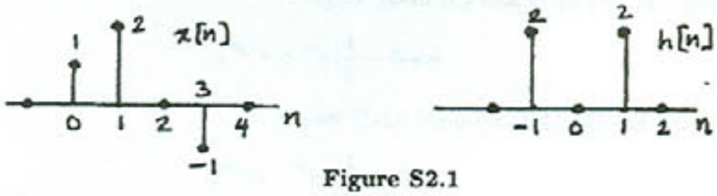


Figure S2.1

From this figure, we can easily see that the above convolution sum reduces to

$$\begin{aligned} y_1[n] &= h[-1]x[n+1] + h[1]x[n-1] \\ &= 2x[n+1] + 2x[n-1] \end{aligned}$$

This gives

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

2.5. The signal $y[n]$ is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

In this case, this summation reduces to

$$y[n] = \sum_{k=0}^{9} x[k]h[n-k] = \sum_{k=0}^{9} h[n-k]$$

From this it is clear that $y[n]$ is a summation of shifted replicas of $h[n]$. Since the last replica will begin at $n = 9$ and $h[n]$ is zero for $n > N$, $y[n]$ is zero for $n > N + 9$. Using this and the fact that $y[14] = 0$, we may conclude that N can *at most* be 4. Furthermore, since $y[4] = 5$, we can conclude that $h[n]$ has *at least* 5 non-zero points. The only value of N which satifies both these conditions is 4.

2.6. From the given information, we have:

$$\begin{aligned} y[n] &= x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} (\frac{1}{3})^{-k} u[-k-1]u[n-k-1] \\ &= \sum_{k=-\infty}^{-1} (\frac{1}{3})^{-k} u[n-k-1] \\ &= \sum_{k=1}^{\infty} (\frac{1}{3})^{k} u[n+k-1] \end{aligned}$$

Replacing k by $p-1$,

$$y[n] = \sum_{p=0}^{\infty} (\frac{1}{3})^{p+1} u[n+p] \qquad \text{(S2.6-1)}$$

For $n \geq 0$ the above equation reduces to,

$$y[n] = \sum_{p=0}^{\infty} (\frac{1}{3})^{p+1} = \frac{1}{3}\frac{1}{1-\frac{1}{3}} = \frac{1}{2}.$$

For $n < 0$ eq. (S2.6-1) reduces to,

$$\begin{aligned} y[n] &= \sum_{p=-n}^{\infty} (\frac{1}{3})^{p+1} = (\frac{1}{3})^{-n+1} \sum_{p=0}^{\infty} (\frac{1}{3})^{p} \\ &= (\frac{1}{3})^{-n+1} \frac{1}{1-\frac{1}{3}} = (\frac{1}{3})^{-n} \frac{1}{2} = \frac{3^n}{2} \end{aligned}$$

Therefore,

$$y[n] = \begin{cases} (3^n/2), & n < 0 \\ (1/2), & n \geq 0 \end{cases}$$

2.8. Using the convolution integral,

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Given that $h(t) = \delta(t+2) + 2\delta(t+1)$, the above integral reduces to

$$x(t) * y(t) = x(t+2) + 2x(t+1)$$

The signals $x(t+2)$ and $2x(t+1)$ are plotted in Figure S2.8.

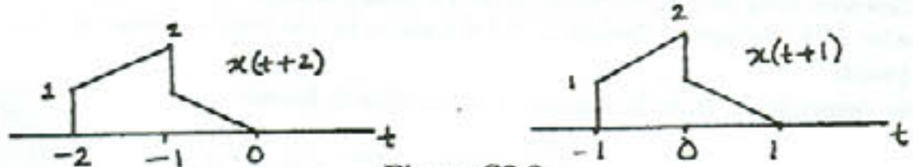


Figure S2.8

Using these plots, we can easily show that

$$y(t) = \begin{cases} t+3, & -2 < t \le -1 \\ t+4, & -1 < t \le 0 \\ 2-2t, & 0 < t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

2.11. (a) From the given information, we see that $h(t)$ is non zero only for $0 \le t \le \infty$. Therefore,

$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-3\tau}(u(t-\tau-3) - u(t-\tau-5))d\tau \end{aligned}$$

We can easily show that $(u(t-\tau-3) - u(t-\tau-5))$ is non zero only in the range $(t-5) < \tau < (t-3)$. Therefore, for $t \le 3$, the above integral evaluates to zero. For $3 < t \le 5$, the above integral is

$$y(t) = \int_{0}^{t-3} e^{-3\tau} d\tau = \frac{1 - e^{-3(t-3)}}{3}$$

For $t > 5$, the integral is

$$y(t) = \int_{t-5}^{t-3} e^{-3\tau} d\tau = \frac{(1 - e^{-6})e^{-3(t-5)}}{3}$$

Therefore, the result of this convolution may be expressed as

$$y(t) = \begin{cases} 0, & -\infty < t \le 3 \\ \frac{1-e^{-3(t-3)}}{3}, & 3 < t \le 5 \\ \frac{(1-e^{-6})e^{-3(t-5)}}{3}, & 5 < t \le \infty \end{cases}$$

2.23. $y(t)$ is sketched in Figure S2.23 for the different values of T.

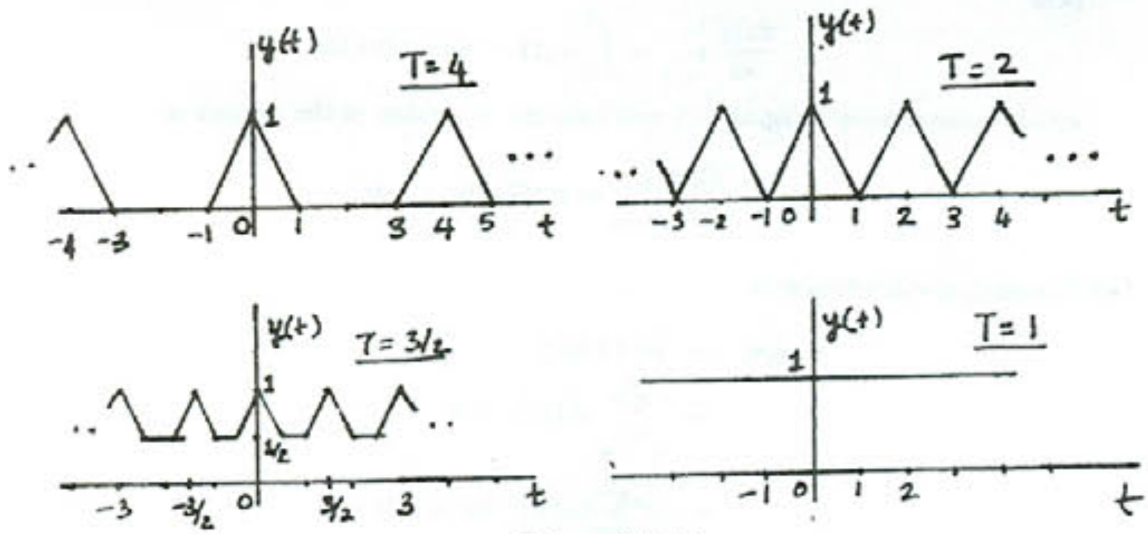
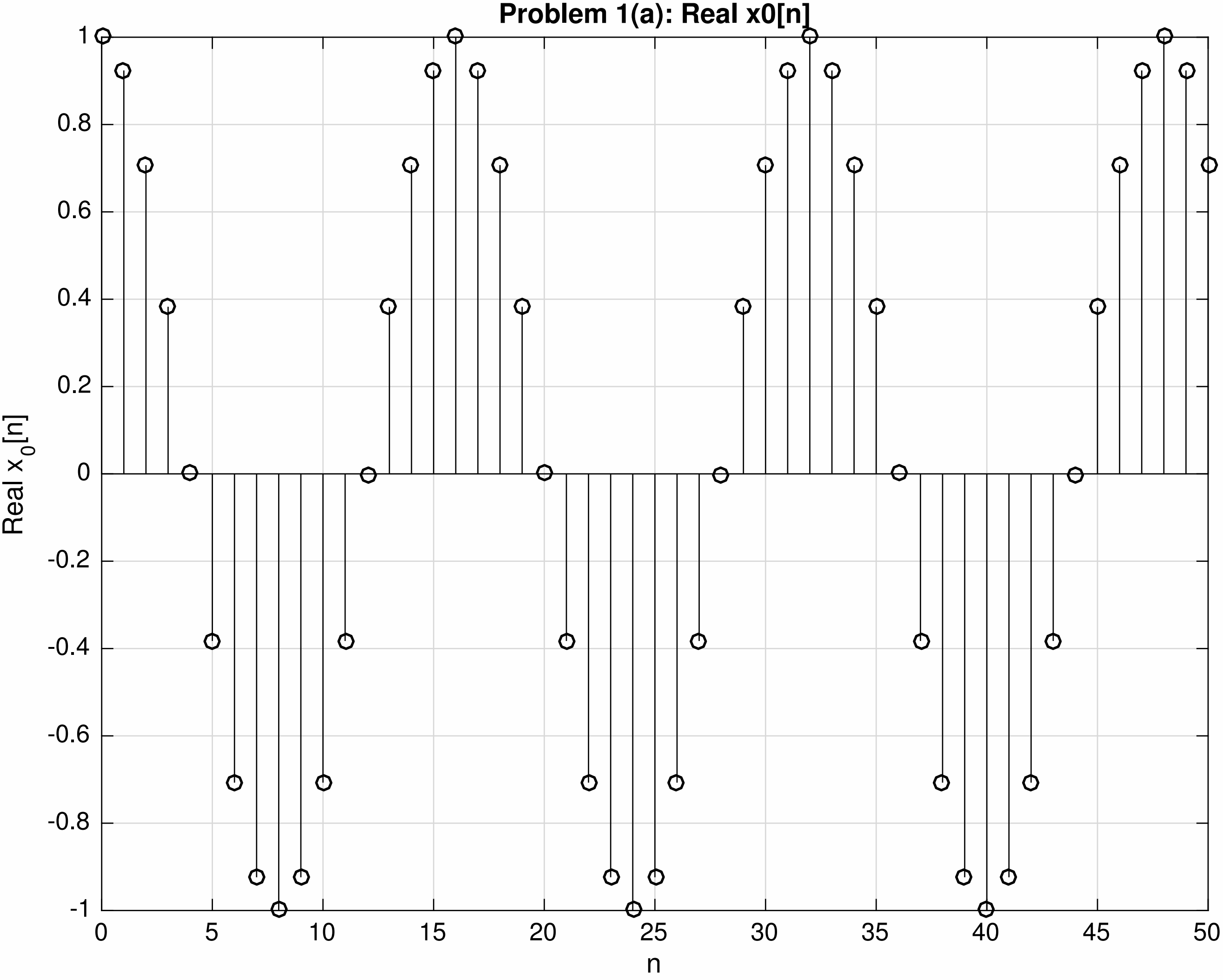
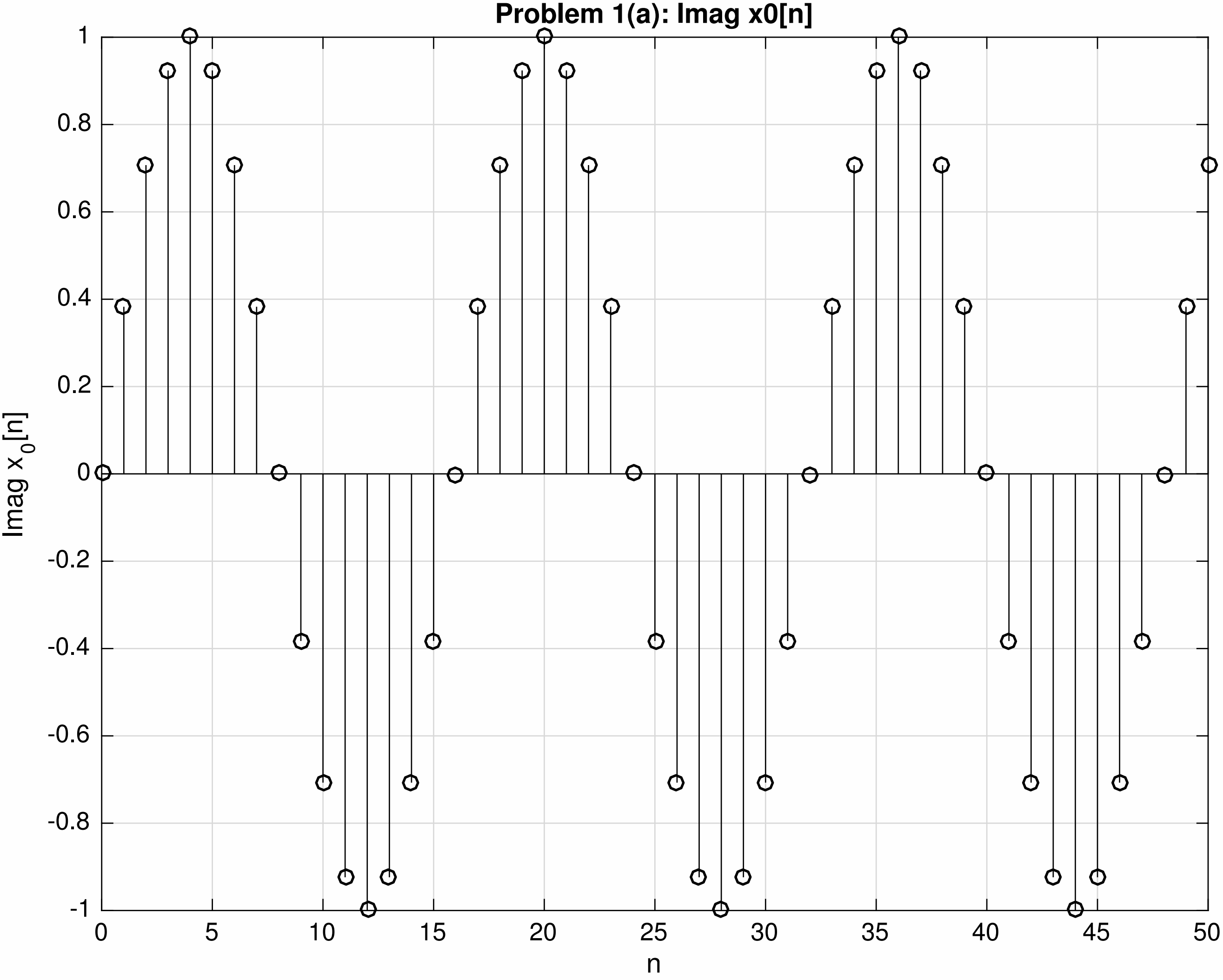


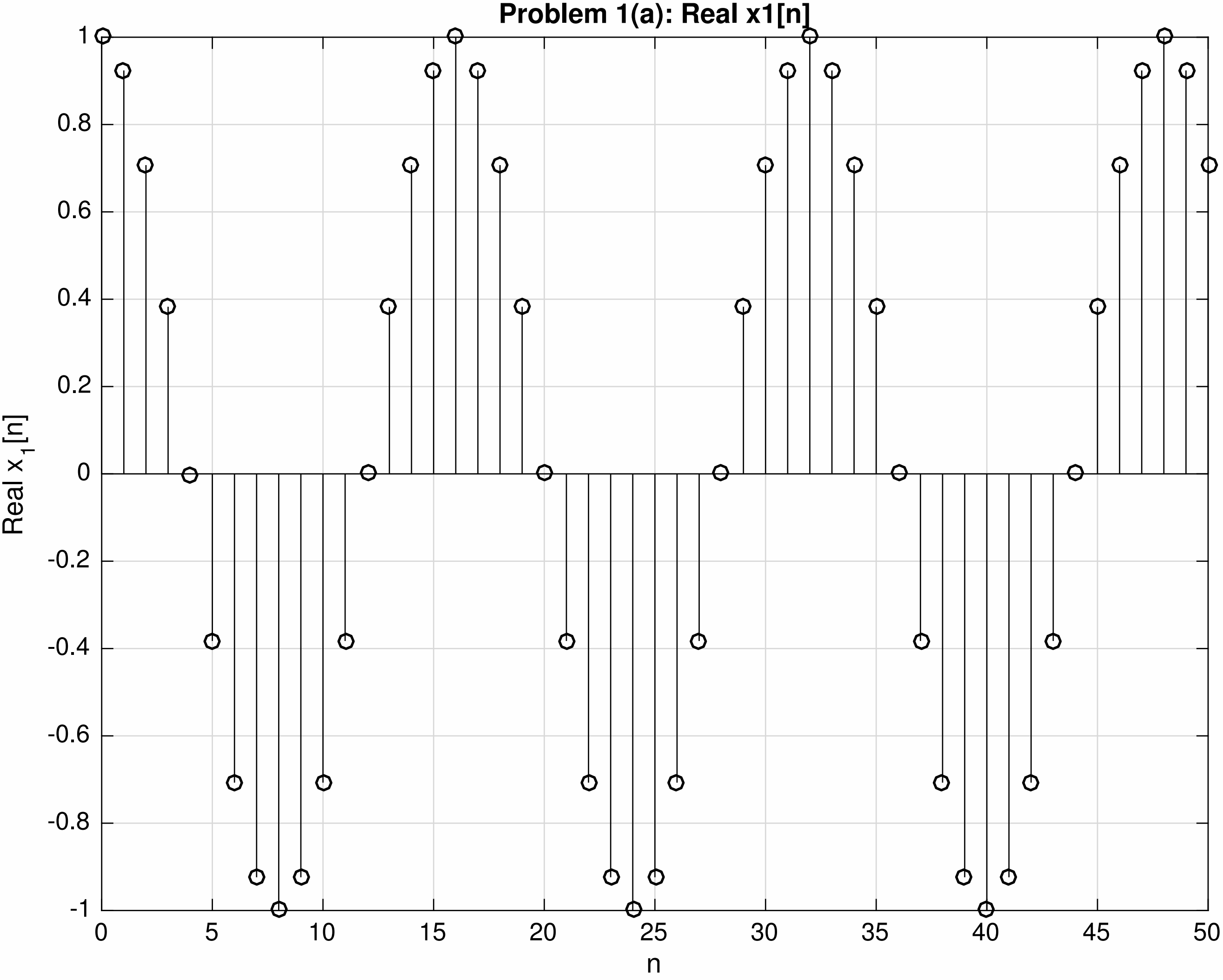
Figure S2.23

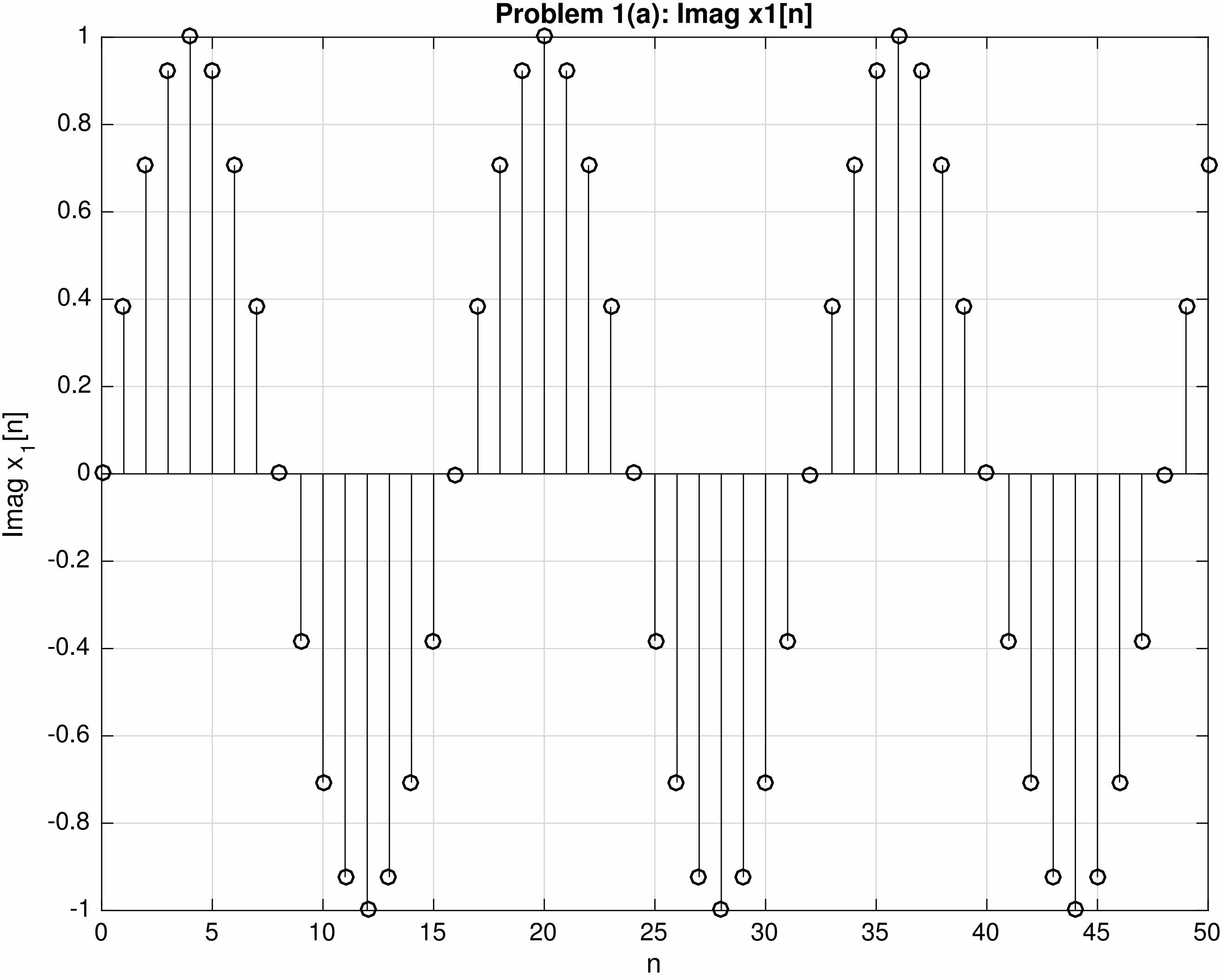


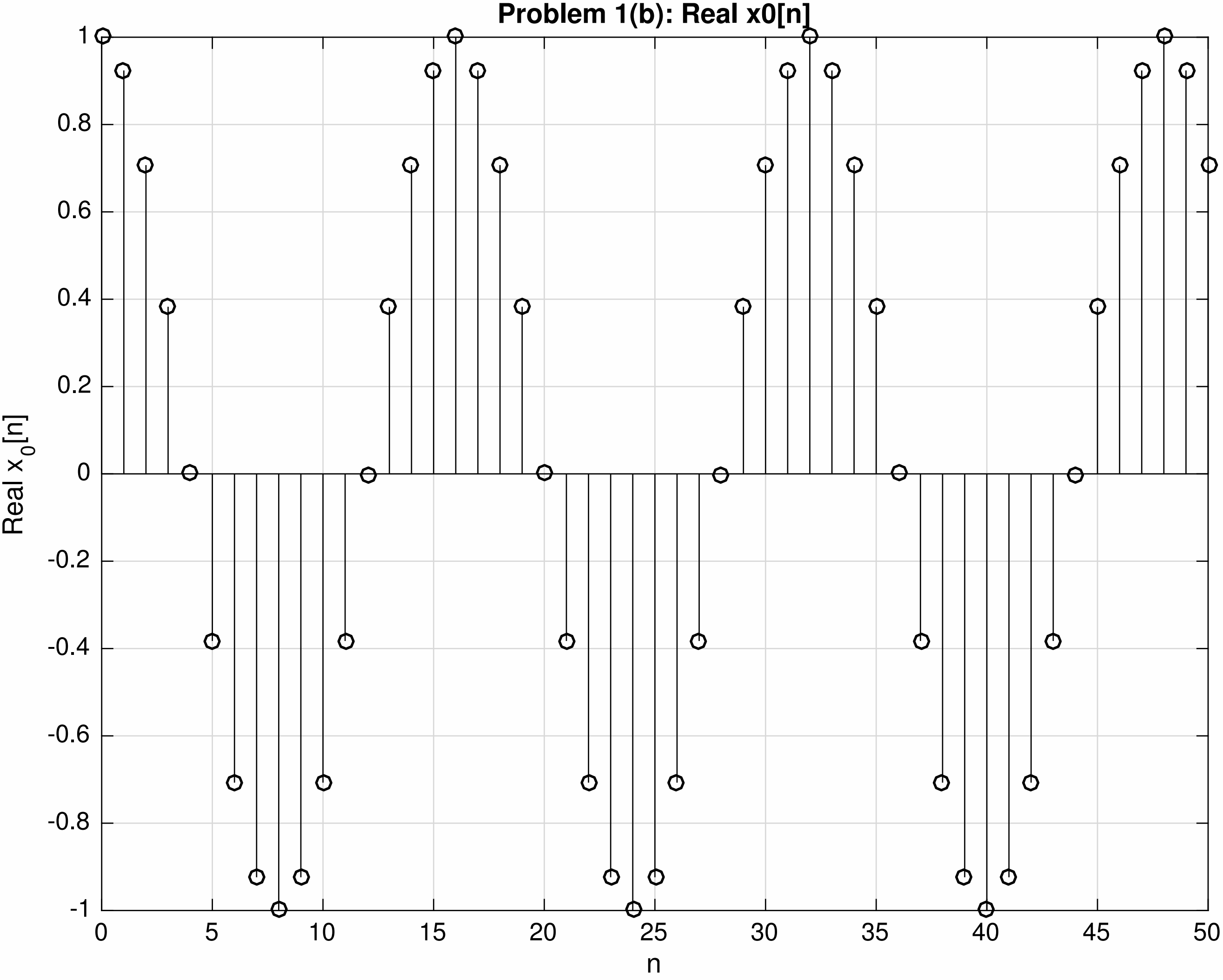
Problem 1(a): Real x0[n]
Real x_0[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50



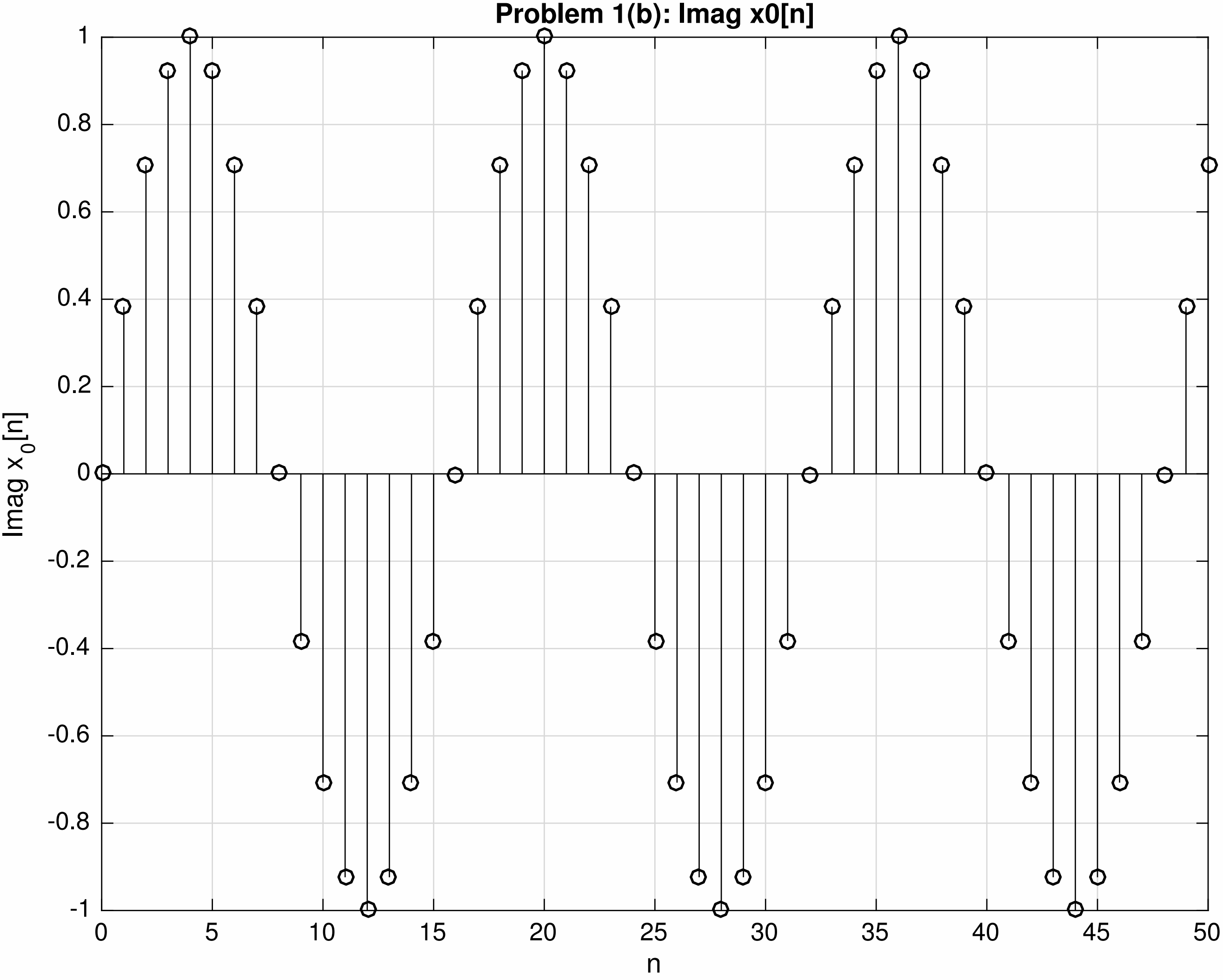
Problem 1(a): Imag x0[n]
Imag $x_0[n]$
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50


Problem 1(a): Real x1[n]
Real $x_1[n]$
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

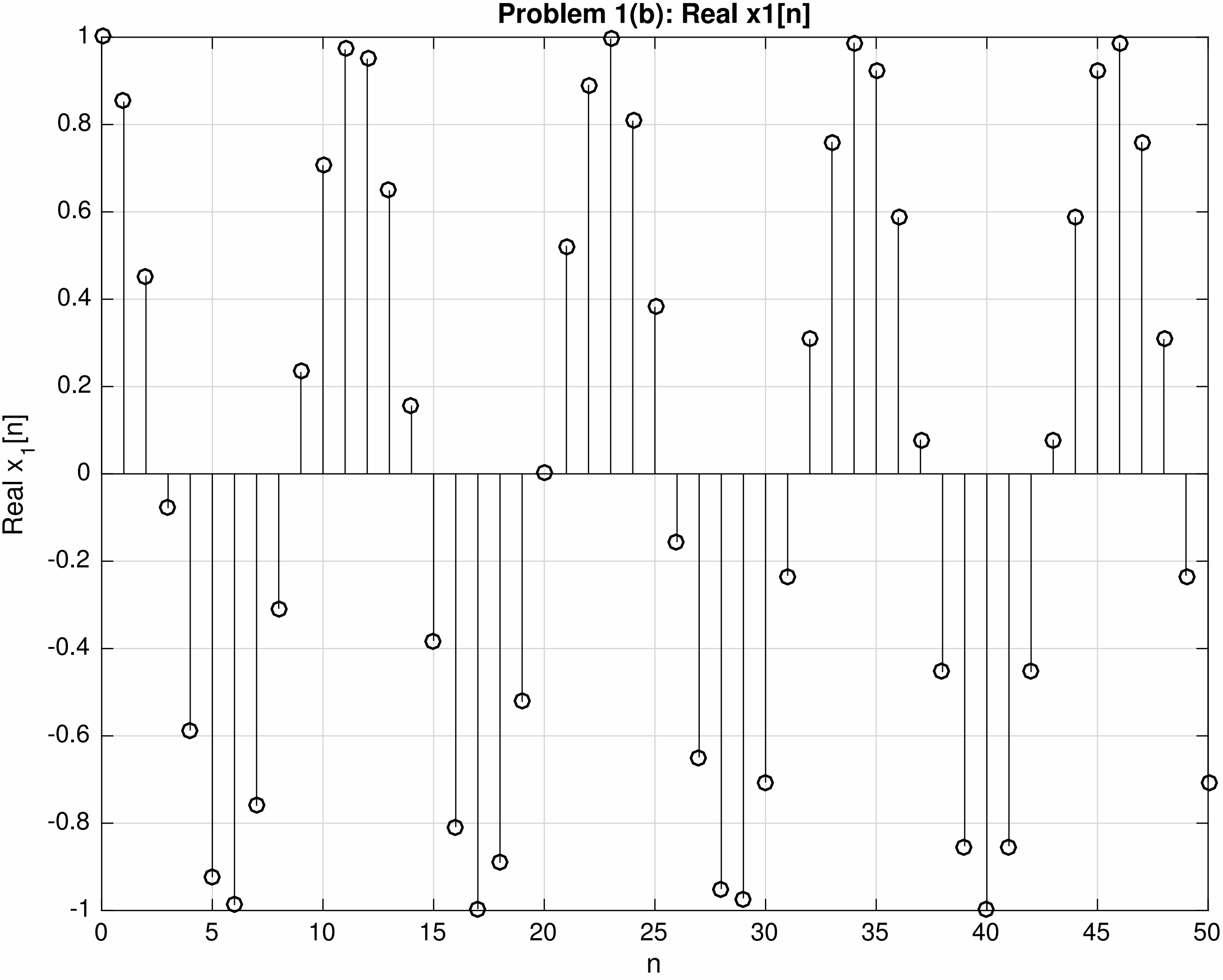

Problem 1(a): Imag x1[n]
Imag x1[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

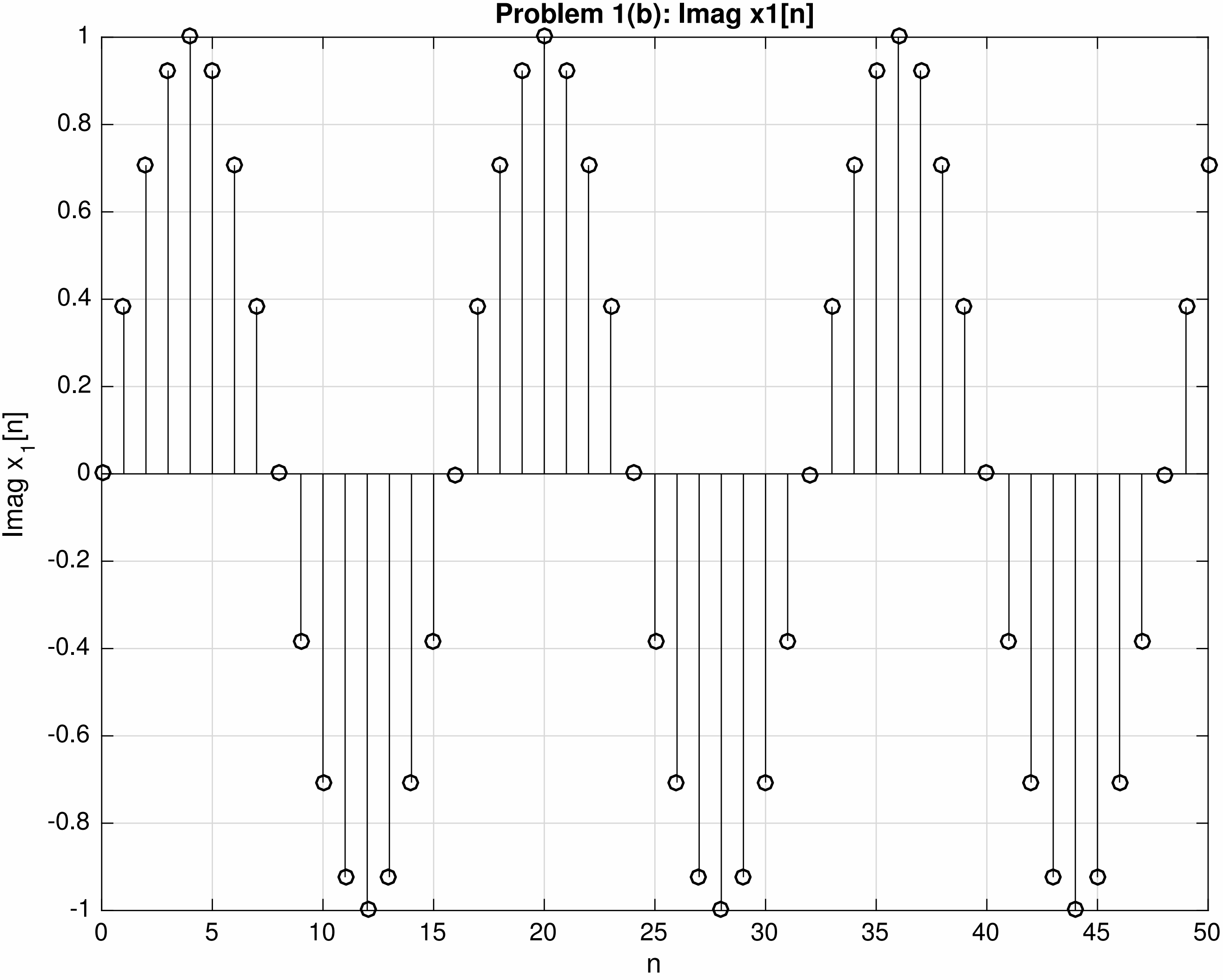


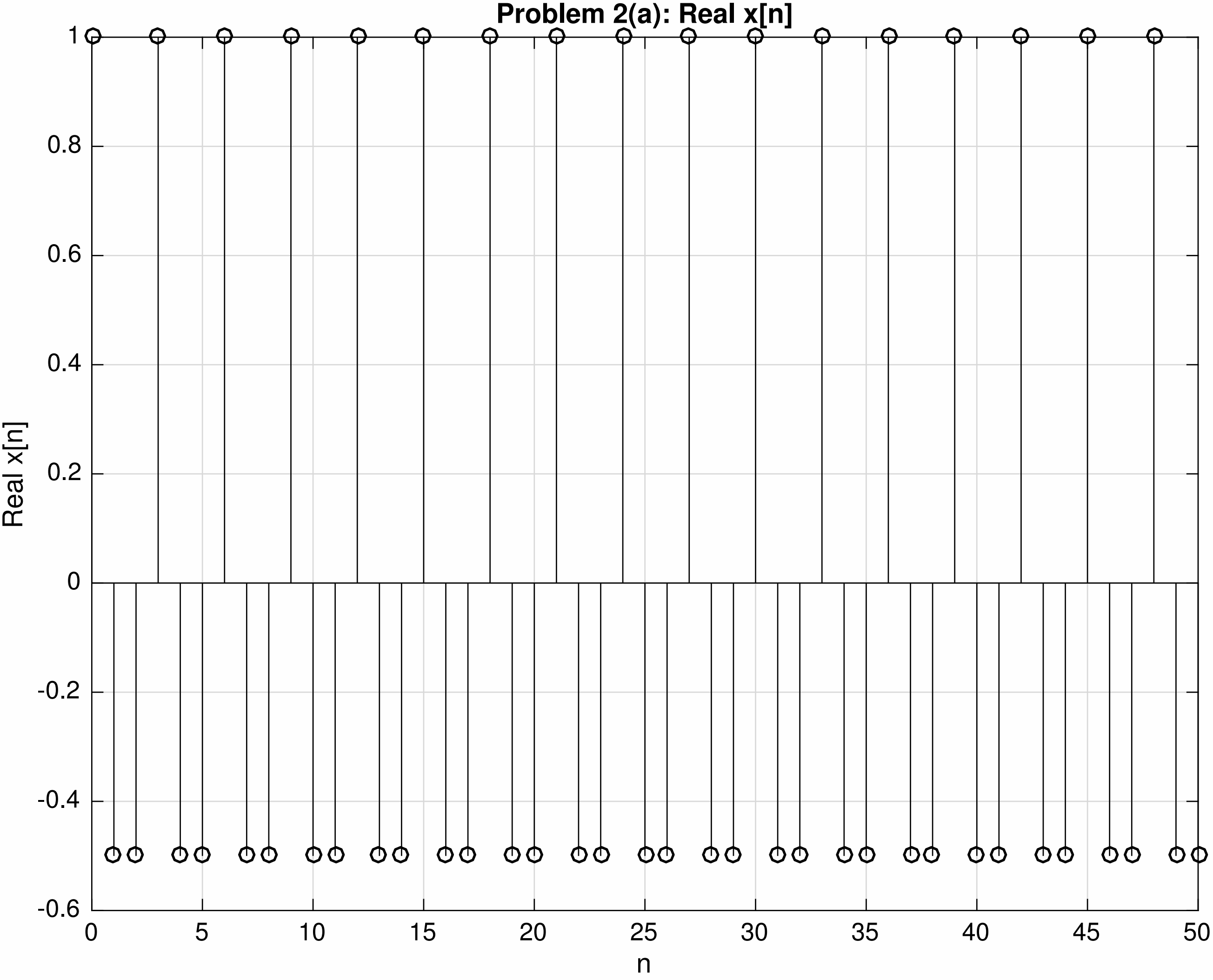
Problem 1(b): Real x0[n]
Real $x_0[n]$
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

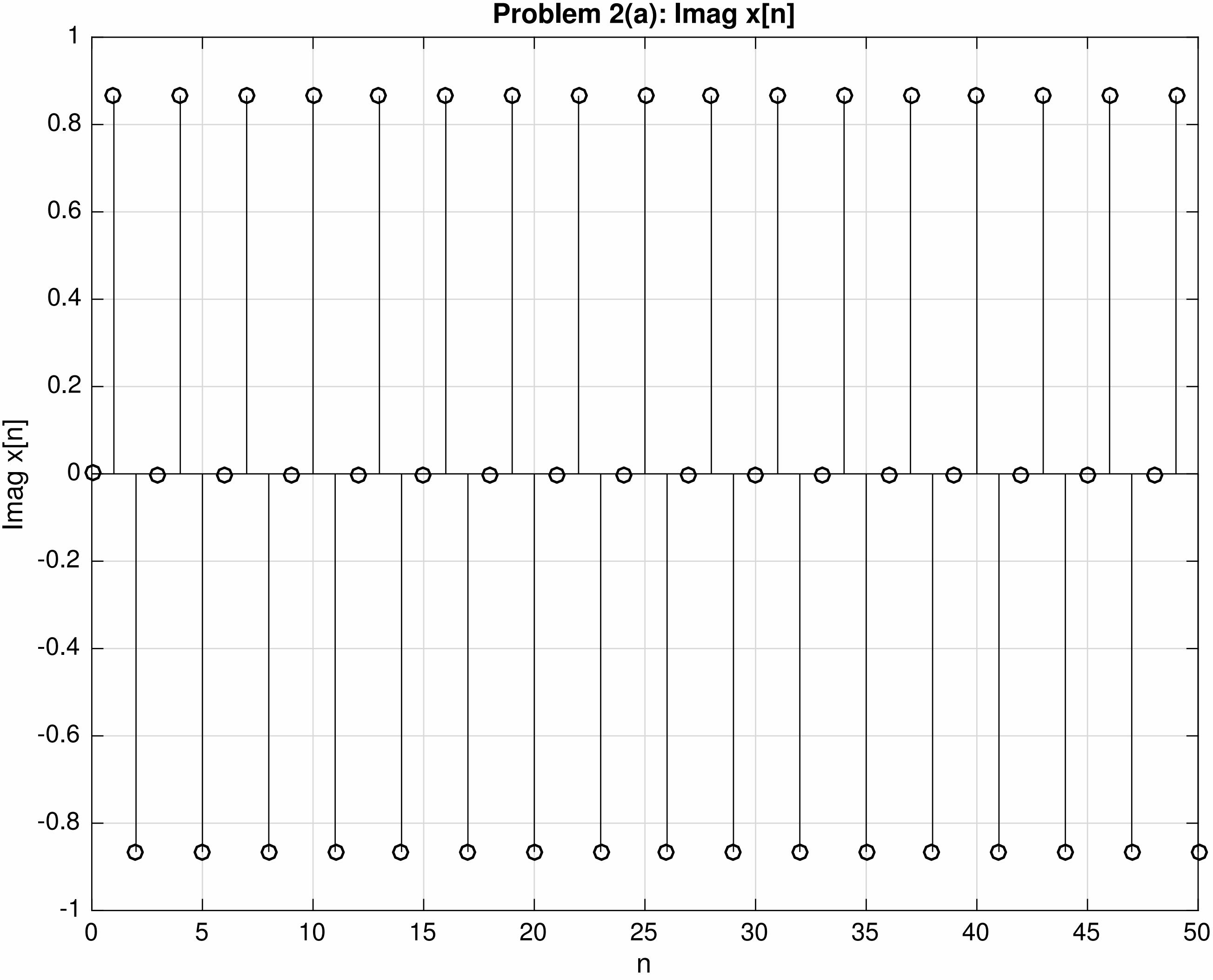


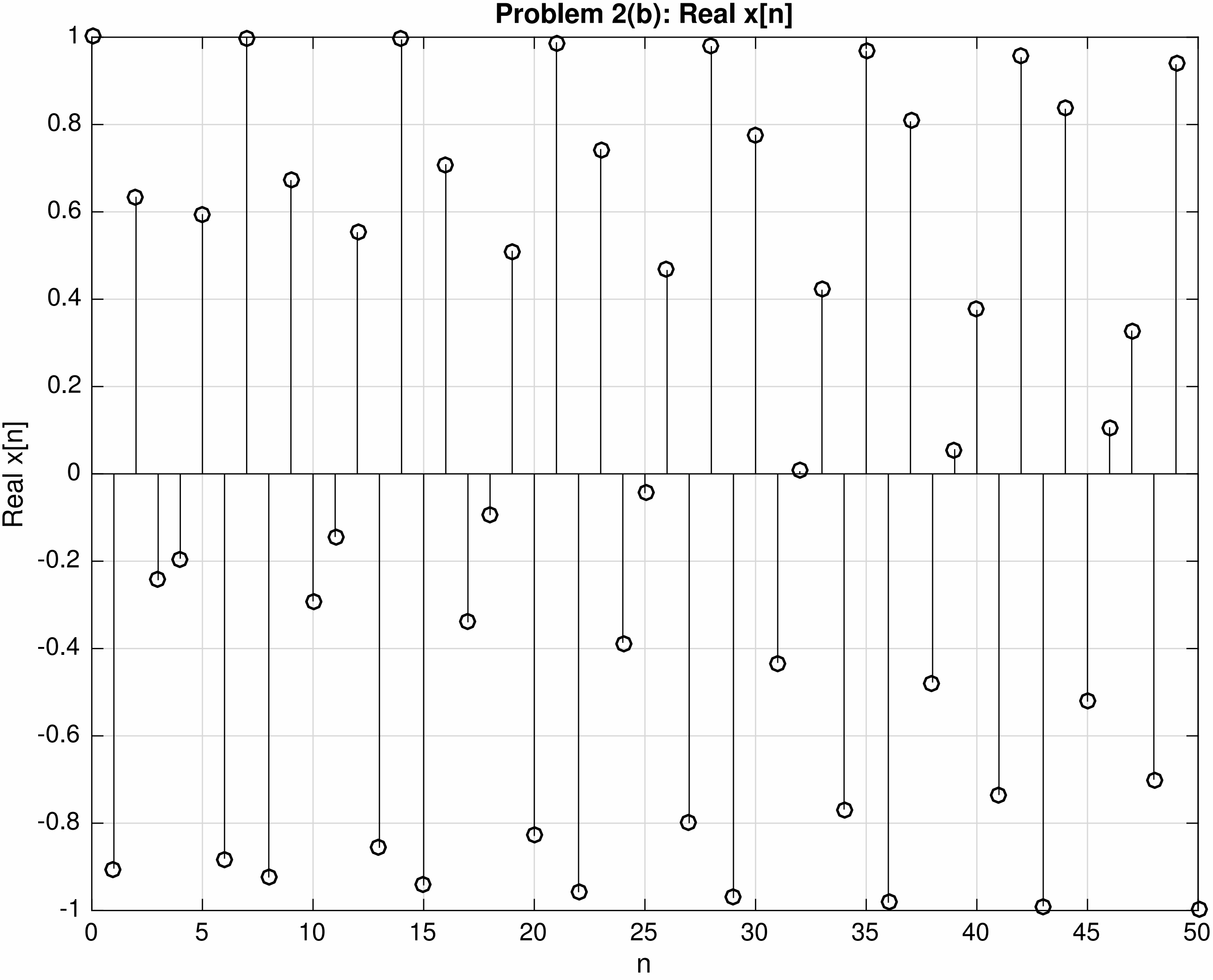
Problem 1(b): Imag x0[n]
Imag x_0[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

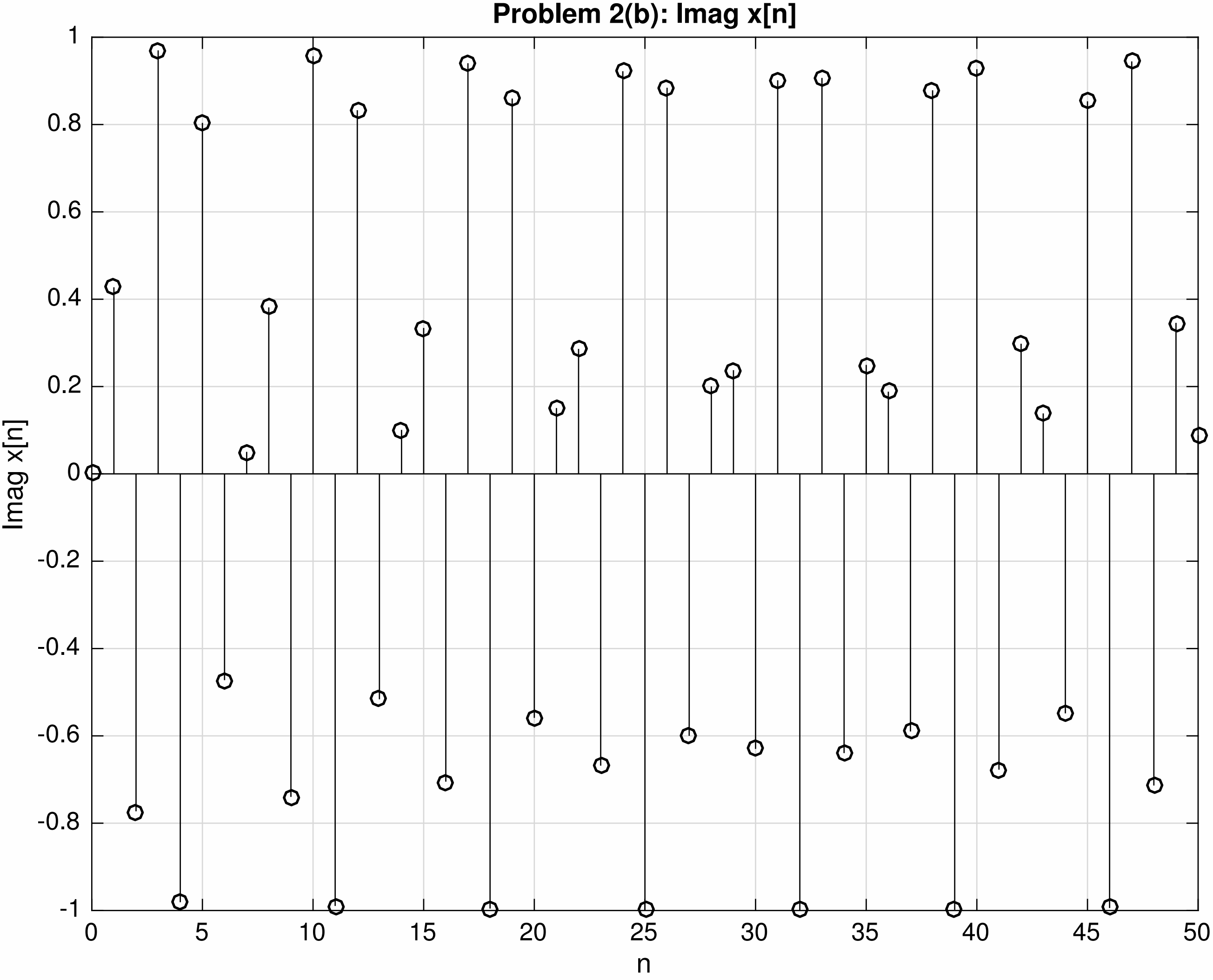

Problem 1(b): Real x1[n]
Real x_1[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

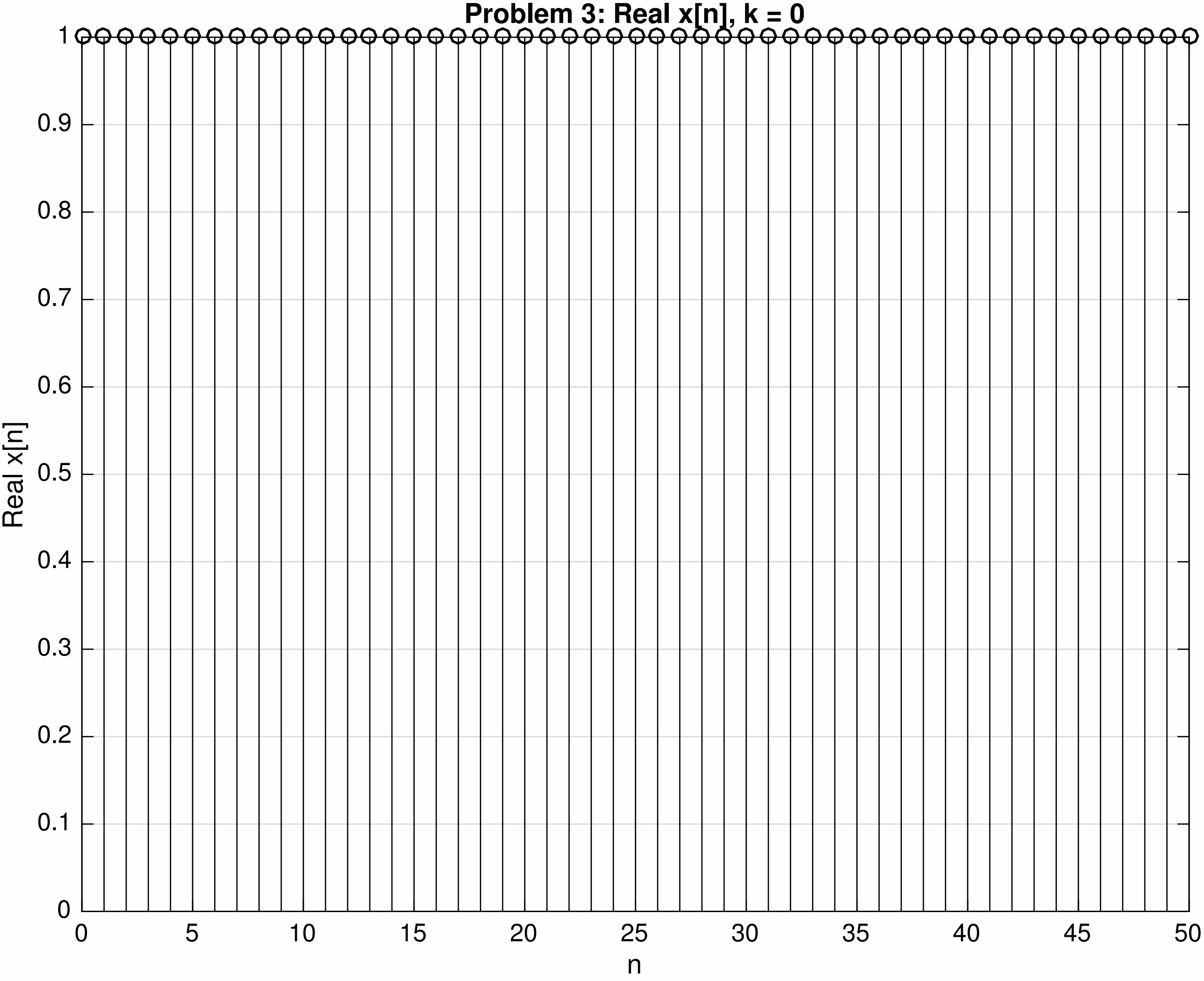

Problem 1(b): Imag x1[n]
Imag x_1[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50


Problem 2(a): Real x[n]
Real x[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
5
10
15
20
25
30
35
40
45
50

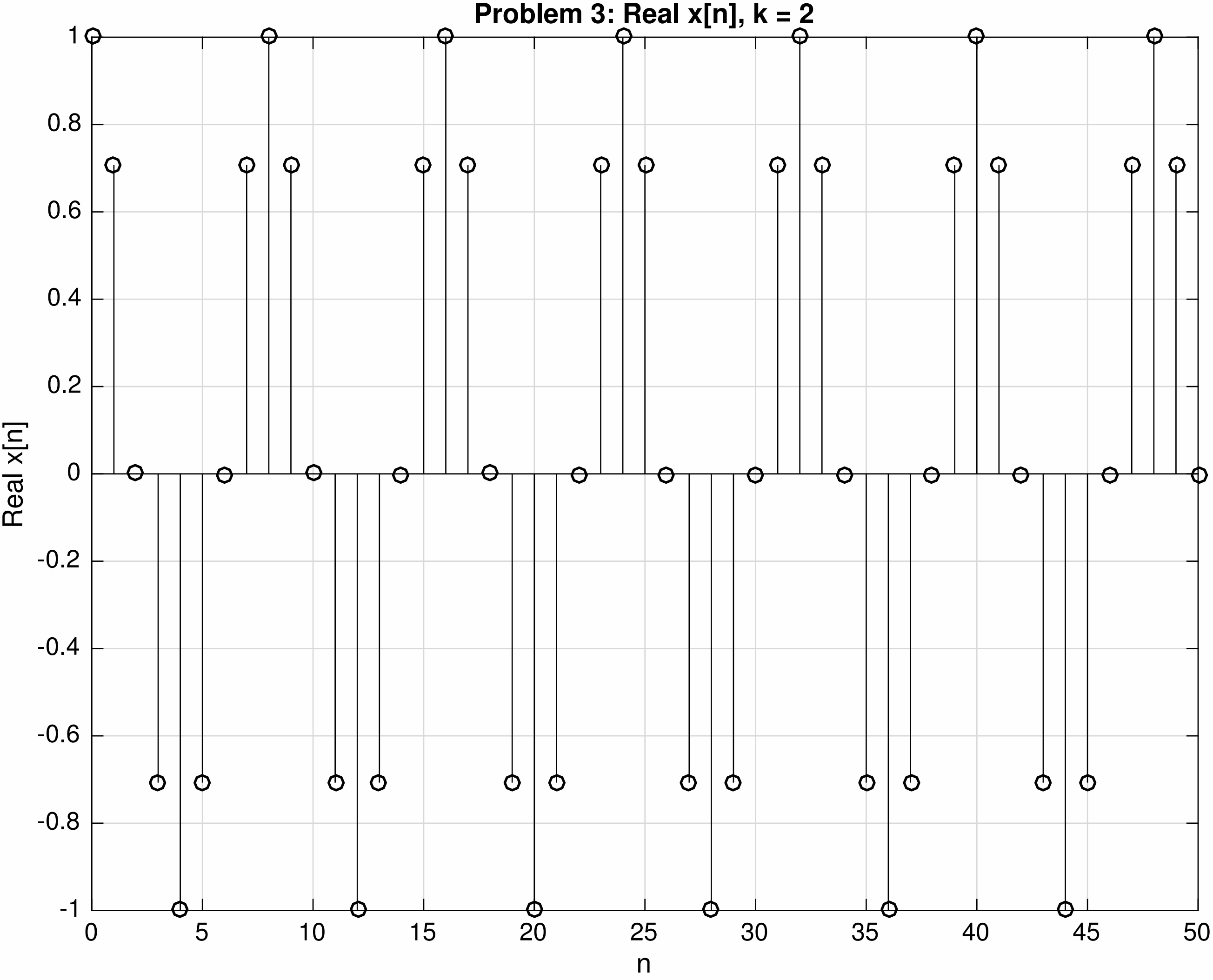

Problem 2(a): Imag x[n]
Imag x[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50


Problem 2(b): Real x[n]
Real x[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

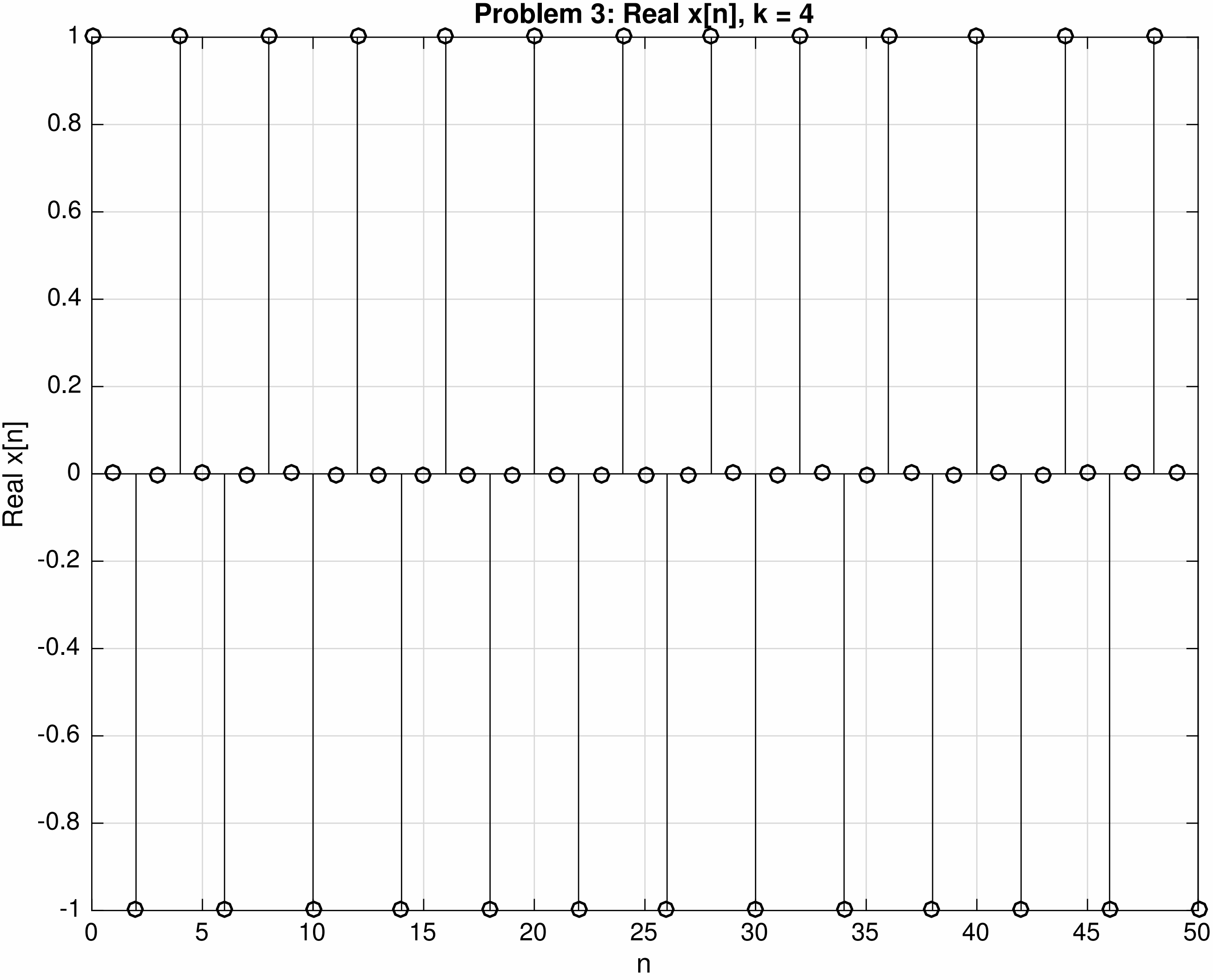

Problem 2(b): Imag x[n]
Imag x[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50



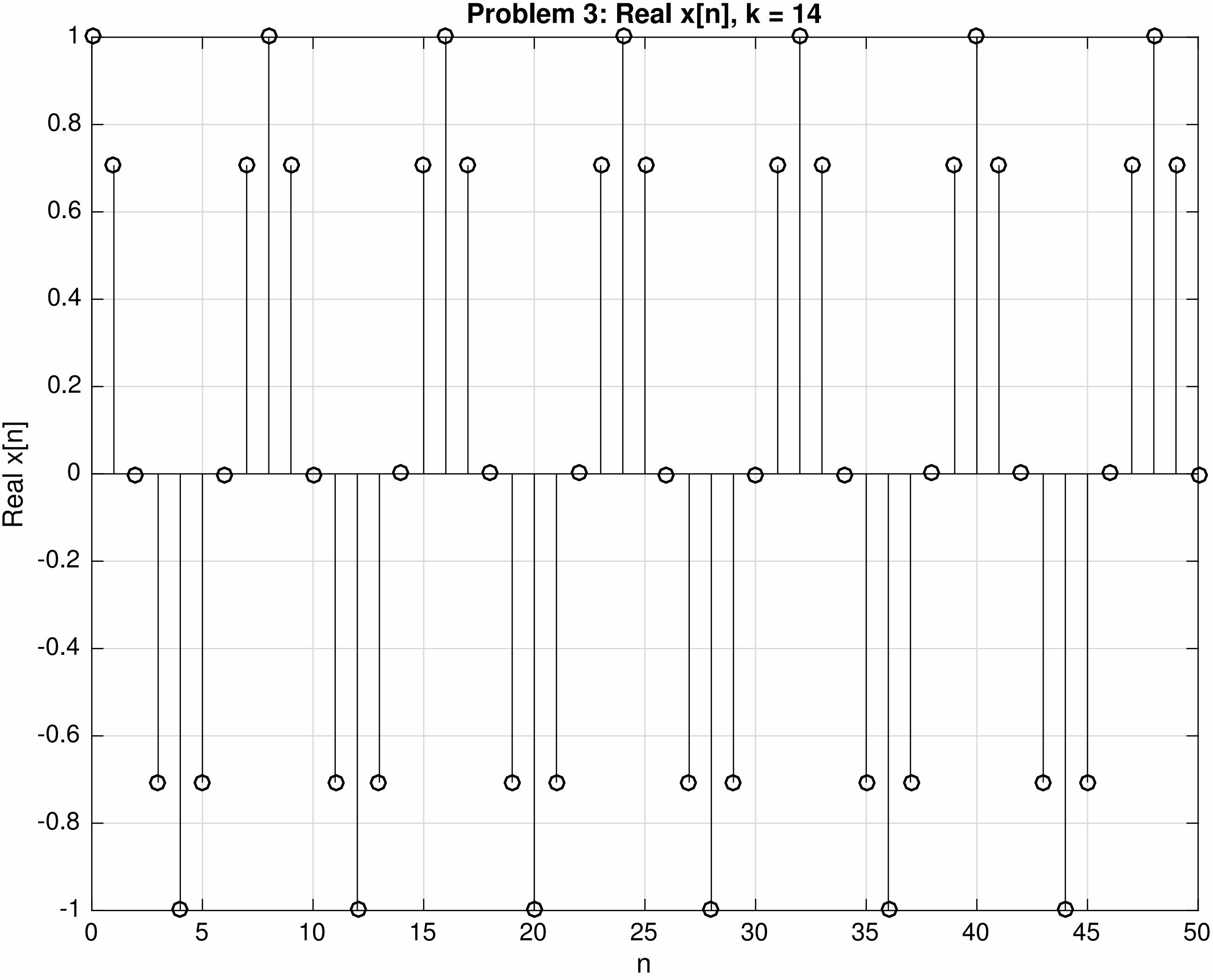
Problem 3: Real x[n], k = 0
Real x[n]
n
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
5
10
15
20
25
30
35
40
45
50

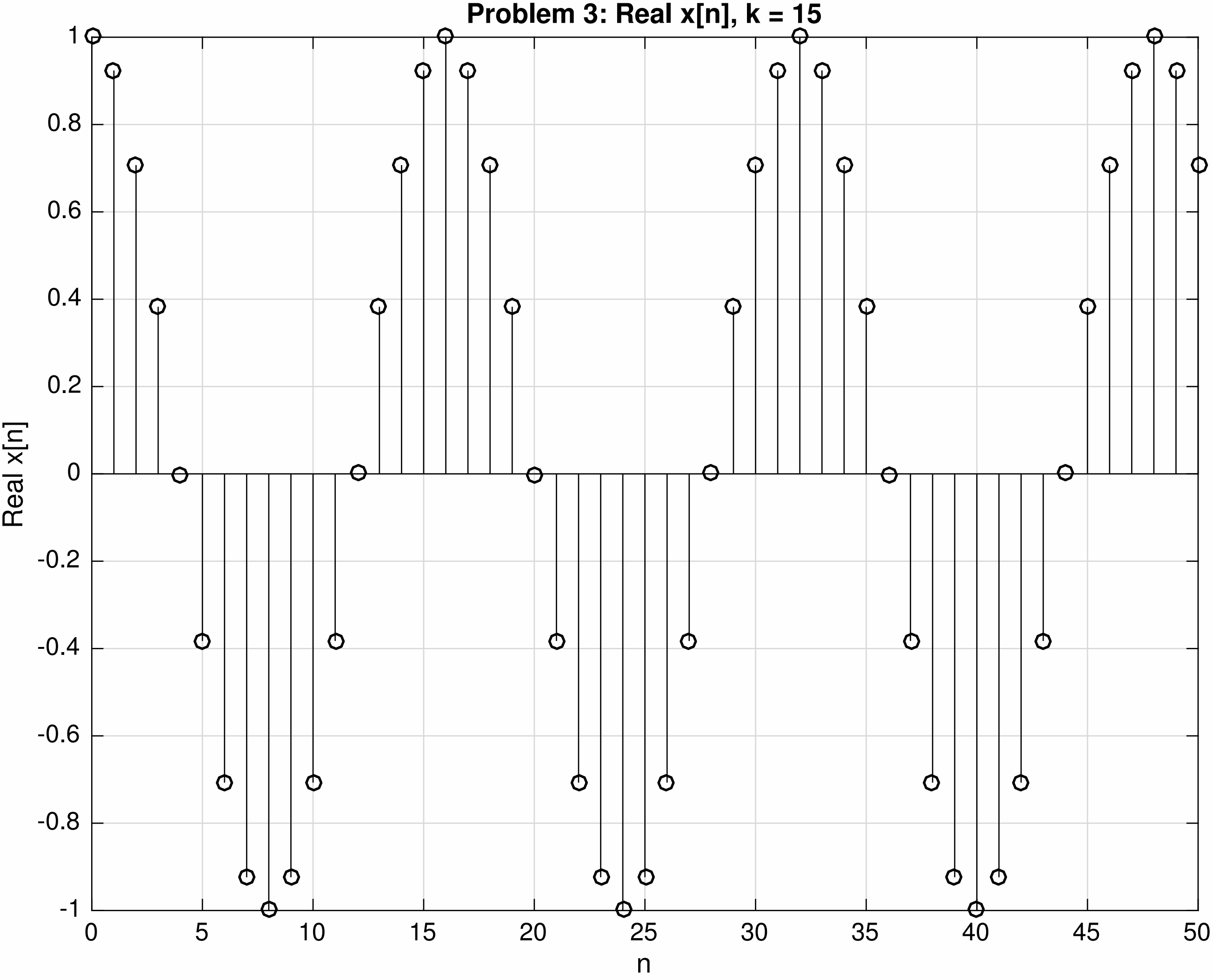


Problem 3: Real x[n], k = 2
Real x[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

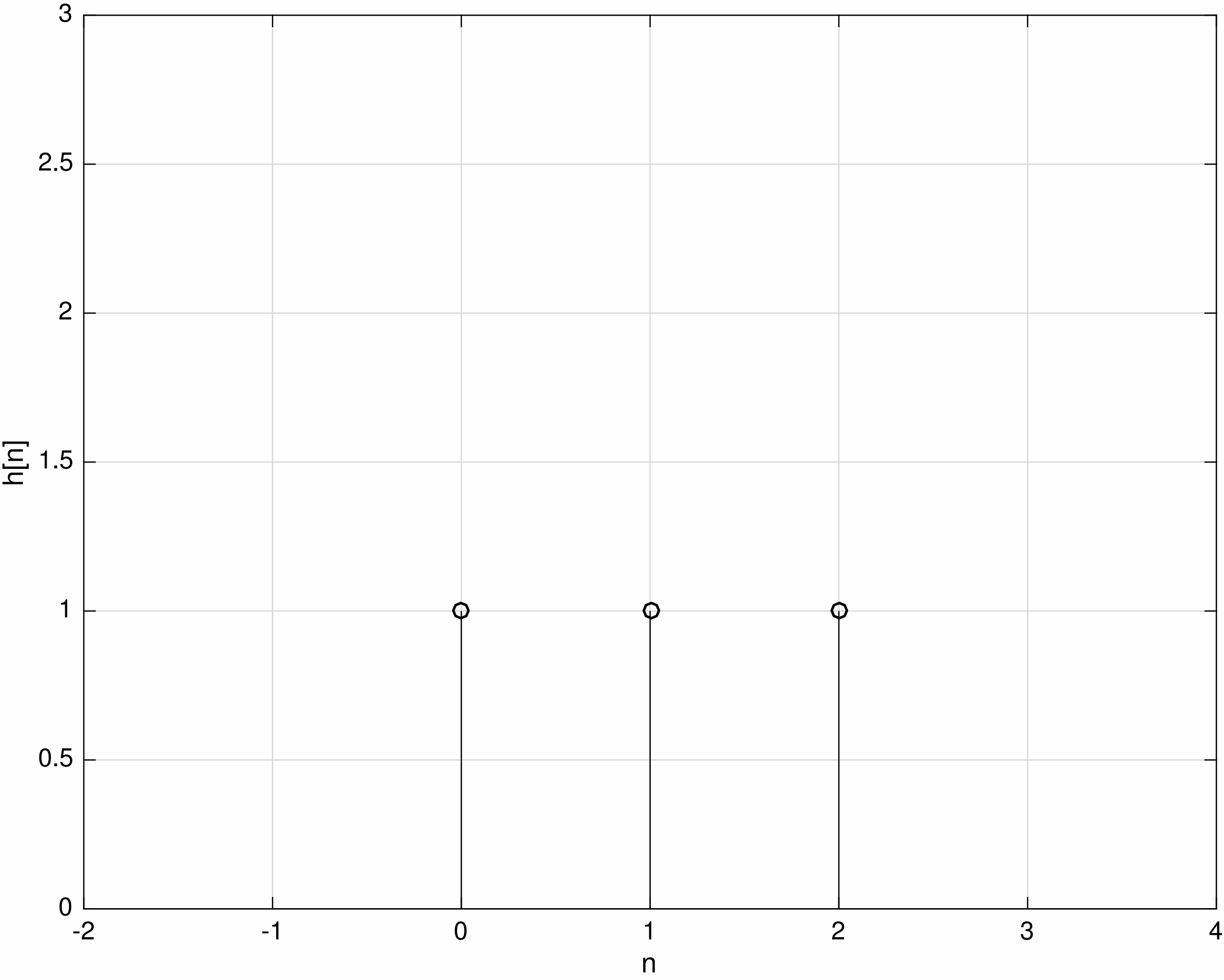


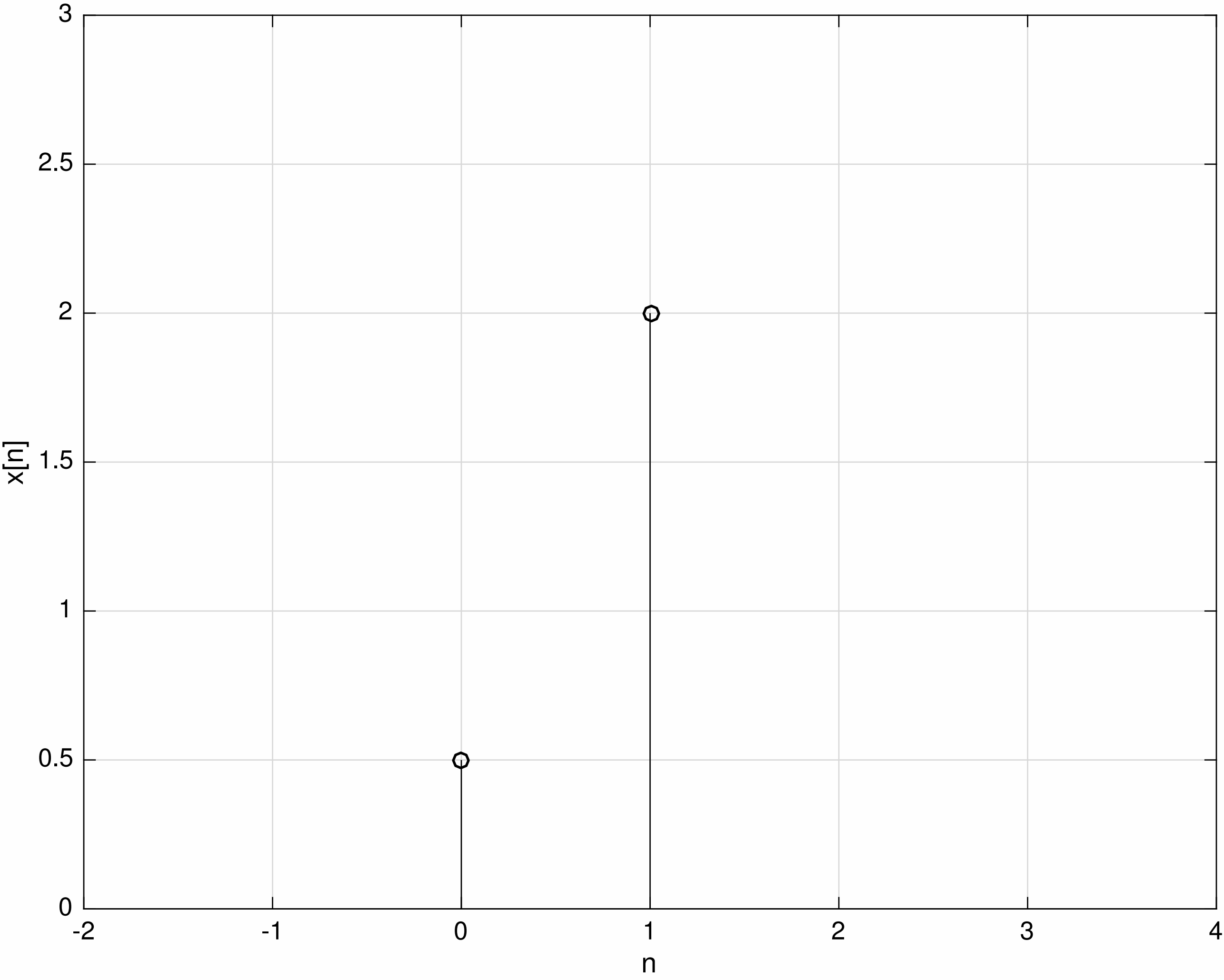
Problem 3: Real x[n], k = 4
Real x[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

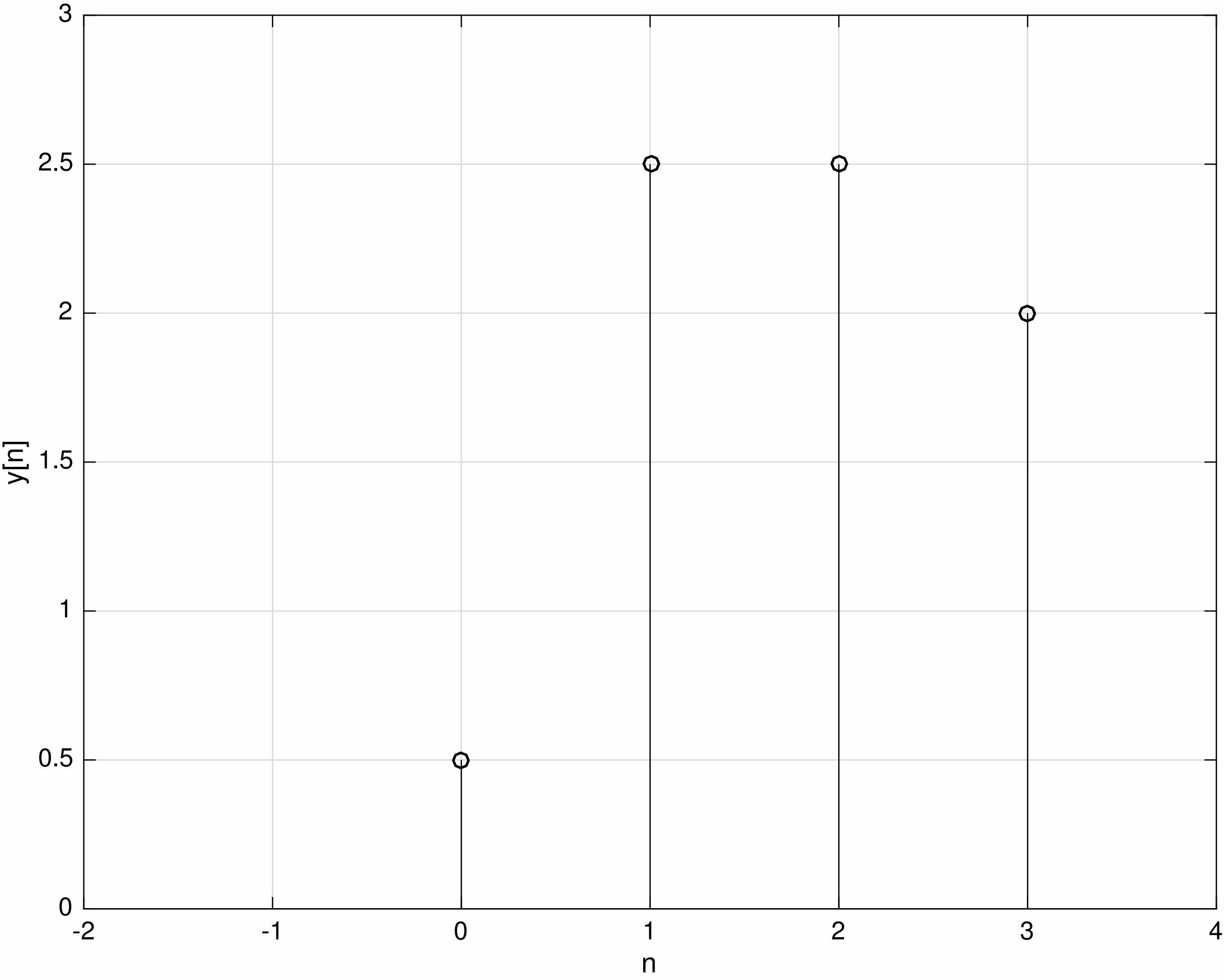

Problem 3: Real x[n], k = 14
Real x[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50



Problem 3: Real x[n], k = 15
Real x[n]
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50


h[n]
n
-2
-1
0
1
2
3
4
0
0.5
1
1.5
2
2.5
3


x[n]
n
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
3
4



y[n]
n
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
3
4