Project #4: Binary Phase-Shift Keying Modem (Receiver)

The key to the detection of the waveforms associated with 0 or 1 will be in the use of a *matched filter* i.e. a filter designed to match to a specific waveform. The idea is that if the filter determines a positive match, a positive number is output; otherwise a negative number is output.

Maximum likelihood detector

Coherent detection of any received digital waveform can be accomplished with a correlating detector also referred to as a *maximum likelihood (ML) detector*. In the figure, the received signal is denoted $r(t)$.

Figure 3.8 (Sklar): Maximum likelihood detector

The BPSK waveform for bit i (0 or 1) is given by

$$s_i(t) = \sqrt{2E/T}\cos(\Omega_c t + \varphi_i)$$

where E is the signal energy per symbol, T is the symbol duration, Ω_c is the carrier frequency, and $\phi_0 = 0$ or $\phi_1 = \pi$ (depending on the transmitted bit). It can be shown that in the binary symbol set case

$$E\{z_0|s_0\} = \sqrt{E}$$
$$E\{z_1|s_0\} = -\sqrt{E}$$
$$E\{z_0|s_1\} = -\sqrt{E}$$
$$E\{z_1|s_1\} = \sqrt{E}$$

where $E\{z_i|s_j\}$ is read as "the expected value of z_i given s_j was transmitted."

The decision stage then selects (decides) which symbol (or bit for the binary case) was transmitted given the largest value of $z_i(nT)$, i.e.

$$\text{symbol} = \begin{cases} s_0, & z_0 > 0 \\ s_1, & z_1 > 0 \end{cases}.$$

Note that in the binary case, the outputs of the ML detectors will always be opposite in sign. Thus we need only one detector since we can base the symbol decision simply on the sign of either $z_0(t)$ or $z_1(t)$.

Sampled Matched Filter

Digital filters can be used to provide the correlator outputs, z_i used in the ML detector as illustrated below.

Figure: Digital filter-based implementation

The N coefficients of the ith matched filter, $\mathbf{c}_i$ are taken as the time-reversed (and shifted by $N - 1$ samples for causality) samples of $\mathbf{s}_i$ (BPSK waveform)

$$c_{i,k} = \begin{cases} s_i[N-1-k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}.$$

It can be shown that the matched filter is optimal in the sense of maximizing the signal-to-noise ratio (SNR) of the received signal, $r(t)$.

Important Note: For the BPSK receiver, we need only build one matched filter (matched to the waveform corresponding to ‘1’ for example) in the detector since coefficients matched to the other waveform (corresponding to ‘0’ for example) and hence filter output will be the negative of the first. We will indicate this single matched filter as $\mathbf{c}$. Note that we do not make bit decisions every sample but rather every symbol period.

The next few examples illustrate the process of matched filtering.

Example (The BPSK Signal): Assume for the BPSK signals we have

$$s_0[n] = \cos(\omega_c n)$$
$$s_1[n] = \cos(\omega_c n + \pi) = -\cos(\omega_c n)$$

where

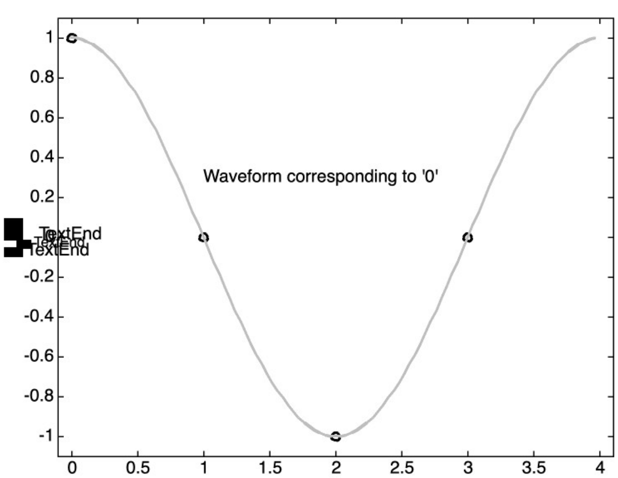
$$\begin{aligned} \omega_c &= 2\pi f_c / f_s \\ &= 2\pi \cdot 2000/8000 \\ &= \pi/2 \end{aligned}$$

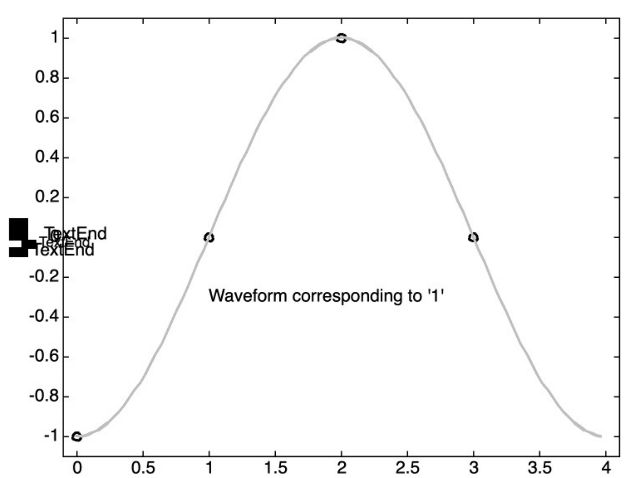
Here $\omega_c = 2\pi f_c / f_s$ is taken as the discrete-time frequency variable. Note that for $f_s = 8000$ and $f_c = 2000$ there will be 4 samples per cycle. Signals for $\mathbf{s}_0(n)$ and $\mathbf{s}_1(n)$ are illustrated below where

$$\mathbf{s}_i[n] = [s_i[n] \quad s_i[n-1] \quad \cdots \quad s_i[n-N+1]]^T$$

and as in the Figure, we have

$$\mathbf{s}_1[3] = [0 \quad 1 \quad 0 \quad -1]^T$$





Note that for R_b = 250 bps, there will be 8 cycles of the waveform for each bit.

Example (The Matched Filter): Let us initialize our length N = 4 matched filter, $\mathbf{c} = [c_0 \quad c_1 \quad \cdots \quad c_{N-1}]^T$ to match against $\mathbf{s}_1[n]$ by time reversing $\mathbf{s}_1[N - 1]$ and shifting by $N - 1$ samples

$$\mathbf{c} = [0 \quad 1 \quad 0 \quad -1]^T .$$

We note that $\mathbf{c} = \mathbf{s}_1[N-1]$ after time reversing and shifting as shown below.

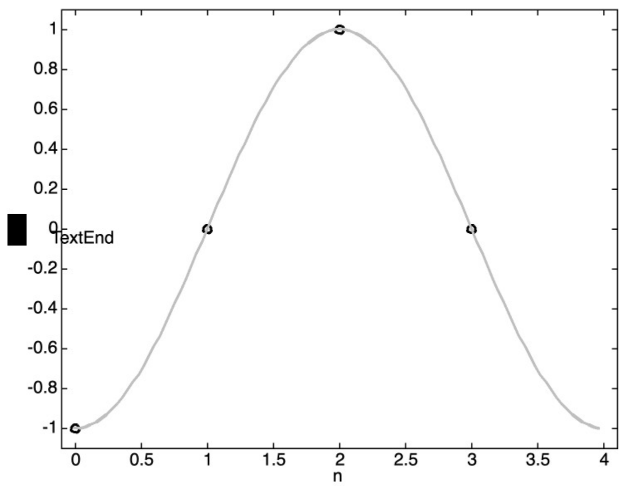


Figure: Sample matched filter, **c** matched against $\mathbf{s}_1(n)$.

Example (The Detection – Part 1): Now suppose we transmit a bit of ‘1’ as in the figure below.

Figure ‘1’ and ‘0’ waveforms (assuming four samples per symbol)

Then our received signal at n = 3 (in the absence of noise) is given by

$$\begin{aligned}\mathbf{r}[3] &= [s[3] \quad s[2] \quad s[1] \quad s[0]]^T \\ &= [0 \quad 1 \quad 0 \quad -1]^T\end{aligned}$$

The output of the matched filter is given by the convolution

$$\begin{aligned} z[3] &= \mathbf{c}^T \mathbf{r}[3] \\ &= \sum_{k=0}^{3} c_k r_k[3] \\ &= 2 \end{aligned}$$

where $r_k[3]$ denotes the k-th element of $\mathbf{r}[3]$. Based on the output of the matched filter, we observe a positive match to $\mathbf{s}_1[n]$ and decide that a '1' was transmitted.

Example (The Detection – Part 2): Now suppose we transmit a '0' during the next symbol period as in the figure above. Then our received signal at $n = 7$ (in the absence of noise) is given by

$$\begin{aligned} \mathbf{r}[7] &= [r[7] \quad r[6] \quad r[5] \quad r[4]]^T \\ &= [0 \quad -1 \quad 0 \quad 1]^T \end{aligned}$$

The output of the matched filter is given by

$$\begin{aligned} z[7] &= \mathbf{c}^T \mathbf{r}[7] \\ &= \sum_{k=0}^{3} c_k r_k[7] \\ &= -2 \end{aligned}$$

where $r_k[7]$ denotes the k-th element of $\mathbf{r}[7]$. Based on the output of the matched filter, we observe a negative match to $\mathbf{s}_1[n]$ and decide that a '0' was transmitted.

Practical Issues

As new samples of the received signal are shifted into $\mathbf{r}[n]$, we must rotate the matched filter coefficients as in the above example in order to maintain match with the '1' bit.

Example: Let us use the signals from the detection examples to transmit (and properly detect) a '1' then a '0' as illustrated below

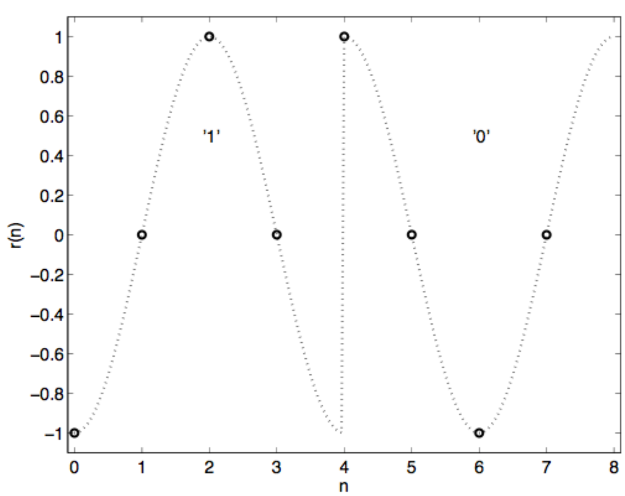


Figure '1' and '0' waveforms (assuming four samples per symbol)

We assume the matched filter, $\mathbf{c}$ has been matched to $\mathbf{s}_1[n]$. The matched filter, received signal vector, and matched filter output at $n = 3$ is given by

$$\mathbf{c}[3] = [0 \quad 1 \quad 0 \quad -1]^T$$
$$\mathbf{r}[3] = [0 \quad 1 \quad 0 \quad -1]^T$$
$$z[3] = 2$$

At this point, we decide a ‘1’ was transmitted since we are positively matched to $\mathbf{s}_1[n]$. Now suppose at the transmitter we begin to transmit a ‘0’. The matched filter, received signal vector, and matched filter output at sample times $n = 4$ and 5 are given by

$$\begin{aligned}\mathbf{c}[4] &= [-1 \quad 0 \quad 1 \quad 0]^T \\ \mathbf{r}[4] &= [1 \quad 0 \quad 1 \quad 0]^T \\ z[4] &= 0\end{aligned}, \qquad \begin{aligned}\mathbf{c}[5] &= [0 \quad -1 \quad 0 \quad 1]^T \\ \mathbf{r}[5] &= [0 \quad 1 \quad 0 \quad 1]^T \\ z[5] &= 0\end{aligned}$$

The small-valued (zero) filter outputs indicate a bit transition is occurring. We continue filtering the received signal and the matched filter, received signal vector, and matched filter output at $n = 6$ and 7 are given by

$$\begin{aligned}\mathbf{c}[6] &= [1 \quad 0 \quad -1 \quad 0]^T \\ \mathbf{r}[6] &= [-1 \quad 0 \quad 1 \quad 0]^T \\ z[6] &= -2\end{aligned}, \qquad \begin{aligned}\mathbf{c}[7] &= [0 \quad 1 \quad 0 \quad -1]^T \\ \mathbf{r}[7] &= [0 \quad -1 \quad 0 \quad 1]^T \\ z[7] &= -2\end{aligned}$$

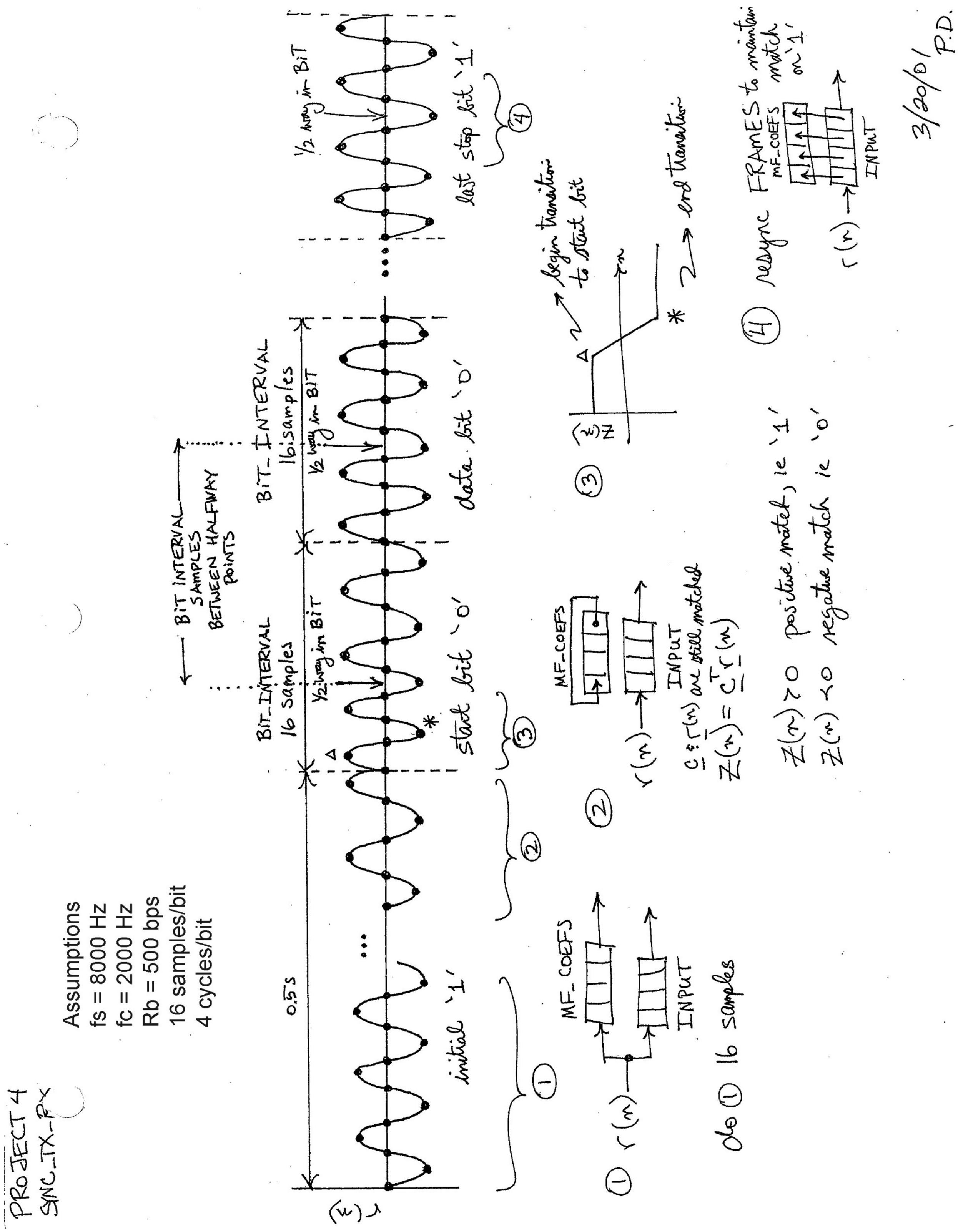
At this point, we decide a ‘0’ was transmitted since we are negatively matched to $\mathbf{s}_1[n]$. Note that we do not make a decision on every sample period since many samples (32 in our case) may represent a single bit. We typically make our decision once we have transitioned completely over to a new bit.

Example: Suppose we have $f_s = 8000$ Hz, $f_c = 2000$ Hz, and a 250 bps data rate. Then each bit duration is 1/250 s and we would have 32 samples per bit period. In this case we would need to decide on a bit every 32 samples. Note also that we have 4 samples/cycle and 8 cycles/bit.

Obviously symbol decisions occur at a rate equal to the data rate. The example previous to the above illustrates that there is a transition period between symbols. Therefore, we want to base our decisions when we are in the “middle” of the symbol interval rather than at the beginning or end since we may be in transition.

Due to inaccuracies in the clocks of both transmitter and receiver, we expect a slight mismatch to occur between the filter coefficients and reference signal after a period of time. Although we plan on making symbol decisions when we are located in the middle of the symbol interval, this slight mismatch will eventually lead us away from the middle of the interval. The easy way around this is to periodically resynchronize the receiver with the transmitter. Since the filter is matched to the ‘1’ signal it will be convenient to reset the coefficients to match on the samples corresponding to the ‘1’ signal after every stop bit.

Figure: Timeline sketch (handout)


PROJECT 4
SYNC_TX_RX
Assumptions
fs = 8000 Hz
fc = 2000 Hz
Rb = 500 bps
16 samples/bit
4 cycles/bit
0.5 s
initial '1'
BIT_INTERVAL
16 samples
start bit '0'
BIT INTERVAL SAMPLES BETWEEN HALFWAY POINTS
data bit '0'
last stop bit '1'
MF_COEFS
INPUT
16 samples
positive match, ie '1'
negative match ie '0'
begin transition to start bit
end transition
resync FRAMES to maintain match on '1'
3/20/01 P.D.