1 Lecture Outline

Reading: Chapter 1 Sampling and Reconstruction

- Overview
- Analog reconstructors
- Ideal reconstruction
- Non-ideal (staircase) reconstruction
- Anti-image postfilters
- Basic components of DSP systems

2 Overview

Figure 1.6.1 illustrates an overview of the components of a DSP system.

Figure 1: Orfanidis p. 54, Fig. 1.7.1 DSP system.

3 Analog reconstructors (1.6)

Reconstruction in DSP is to create a CT signal from samples (DT signal). Therefore we must find a way to "fill in the gaps" or interpolate between samples. Clearly, any reasonable way to interpolate will result in some sort of reconstruction.

Example: One easy method of reconstruction is to simply hold the current sample value constant until the next sample. This results in the staircase reconstructor.

Figure 2: Orfanidis p. 42, Fig. 1.6.1 Staircase reconstructor.

Interpolating between the samples amounts to smoothing or lowpass (LP) filtering of the signal. Thus any reconstructor (not just the staircase reconstructor) will be viewed as a CT lowpass filter (LPF).

Figure 3: Orfanidis p. 42, Fig. 1.6.2 Analog reconstructor as a lowpass filter.

As we have shown earlier we can describe the "sampled input" as (CT signal with zero between samples)

$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT)\delta(t-nT). \tag{1}$$

Given this input and the reconstructor, $h(t)$ we can compute the output with a convolution

$$\begin{aligned} y_a(t) &= \hat{y}(t) \star h(t) \\ &= \int_{-\infty}^{\infty} h(t-\tau)\hat{y}(\tau)d\tau. \end{aligned} \tag{2}$$

Therefore, substituting (1) into (2) and applying the sifting property of the FT we arrive at

$$\begin{aligned} y_a(t) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(t-\tau)y(nT)\delta(\tau-nT)d\tau \quad \text{; substitution of ``sampled signal'' expression} \\ &= \sum_{n=-\infty}^{\infty} y(nT)h(t-nT) \quad \text{; sifting property} \end{aligned} \tag{3}$$

From this we see that a copy of $h(t)$ is attached to each sample $y(nT)$ and each of these terms is superimposed to form the resulting reconstructed CT signal.

The convolution in (2) can be rewritten as a multiplication in the frequency domain

$$Y_a(f) = H(f)\hat{Y}(f) \tag{4}$$

where $\hat{Y}(f)$ is the replicated spectrum, i.e. spectrum of the "sampled signal"

$$\hat{Y}(f) = \frac{1}{T}\sum_{m=-\infty}^{\infty} Y(f-mf_s). \tag{5}$$

Let's now examine a couple of reconstructors, i.e. specific choices for $h(t) \leftrightarrow H(f)$.

3.1 Ideal Reconstructors (1.6.1)

3.1.1 Frequency Domain View

For perfect or ideal reconstruction one must require that $Y_a(f)$ be identical to the original analog spectrum $Y(f)$. If the spectrum $Y(f)$ is bandlimited and its replicas do not overlap, then within the Nyquist interval, $T\hat{Y}(f)$ will agree with $Y(f)$ in accordance with (5), that is,

$$\hat{Y}(f) = \frac{1}{T}Y(f), \quad \text{for } \frac{-f_s}{2} \leq f \leq \frac{f_s}{2} \tag{6}$$

Figure 4: Orfanidis p. 44, Fig. 1.6.3 Ideal reconstructor in frequency domain.

The ideal reconstruction filter $H(f)$ is therefore an ideal lowpass filter with cutoff $f_s/2$, defined as follows:

$$H(f) = \begin{cases} T, & |f| \leq f_s/2 \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Within the Nyquist interval, this leads to

$$\begin{aligned} Y_a(f) &= H(f)\hat{Y}(f) \\ &= T\frac{1}{T}Y(f) \\ &= Y(f). \end{aligned} \tag{8}$$

Outside the Nyquist interval, $H(f)$ is 0 and so $Y_a(f) = Y(f)$ for all f. Therefore the reconstructed signal is identical to the original signal that was sampled.

3.1.2 Time Domain View

In order to understand the ideal reconstructor in the time-domain, we can obtain the impulse response of the ideal reconstructor by taking the IFT of $H(f)$

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}df \\ &= \int_{-f_s/2}^{f_s/2} Te^{j2\pi ft}df \\ &= \frac{\sin(\pi t/T)}{\pi t/T} \\ &= \text{sinc}\left(\pi t/T\right); \quad \text{sinc}(x) \equiv \sin(x)/x \end{aligned} \tag{9}$$

and is shown in Fig. 1.6.4.

Figure 5: Orfanidis p. 44, Fig. 1.6.4 Impulse response of ideal reconstructor.

So we would have as our reconstructed waveform

$$\begin{aligned} y_a(t) &= \sum_{n=-\infty}^{\infty} y(nT)h(t-nT) \\ &= \sum_{n=-\infty}^{\infty} y(nT)\text{sinc}\left[\frac{\pi}{T}(t-nT)\right] \end{aligned} \tag{10}$$

Figure 6: Ideal reconstruction time domain.

Unfortunately, the impulse response is not causal and so this filter is not physically realizable.

3.2 Non-Ideal (Staircase) Reconstructors (1.6.2)

The staircase reconstructor shown in Fig. 1.6.1 is the simplest and most widely used reconstructor in practice. It generates a staircase approximation to the original signal. Thus, $h(t)$ is given by

$$\begin{aligned} h(t) &= u(t) - u(t-T) \\ &= \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{11}$$

or equivalently given by the frequency response

$$\begin{aligned} H(f) &= \frac{1}{j2\pi f}\left(1 - e^{-j2\pi fT}\right) \\ &= T\frac{\sin(\pi fT)}{\pi fT}e^{-j\pi fT} \\ &= T\text{sinc}(\pi fT)e^{-j\pi fT} \end{aligned} \tag{12}$$

Figure 7: Orfanidis p. 45, Fig. 1.6.5 Impulse response and magnitude response of staircase reconstructor.

Of course this filter will yield the staircase approximation to the original signal given the samples. The effect of this filter on the spectrum of the reconstructed signal is

Figure 8: Orfanidis p. 46, Fig. 1.6.6 Frequency response of staircase reconstructor.

4 Anti-Image Postfilters

After staircase reconstruction, the surviving images or pieces of images can be removed with further LP filtering or anti-image postfiltering.

5 Basic Components of DSP Systems (1.7)

The components of typical DSP system and the spectra of intermediate signals are shown in Fig. 1.7.1.

Figure 9: Orfanidis p. 54, Fig. 1.7.1 Components of typical DSP system.