1 Lecture Outline

Reading: Chapter 12 Interpolation, Decimation, and Oversampling

This lecture will cover the following topics

- Decimation and Oversampling (Section 12.5)
- Sampling Rate Converters (Section 12.6)

2 Decimation and Oversampling (Section 12.5)

Decimation by an integer factor L is the reverse of interpolation, that is, decreasing the sampling rate or *downsampling* from the high rate f'_s to the lower rate $f_s = f'_s/L$. Formally, the downsampled signal is defined in terms of the slow time scale as

$$x_{\text{down}}(n) = x'(n')|_{n'=nL} = x'(nL) \tag{1}$$

where n is the sample index for the signal at f_s and n' is the sample index for the signal at f'_s. For the ideal situation depicted in Fig. 12.5.1, the downsampled signal $x_{\text{down}}(n)$ coincides with the low-rate signal $x(n)$ that would have been obtained had the analog signal been resampled at the lower rate $f_s = f'_s/L$.

The gaps in the input spectrum $X'(f)$ are necessary to guarantee this equality. Dropping the sampling rate by a factor of L, shrinks the Nyquist interval $[-f'_s/2, f'_s/2]$ by a factor of L to the new interval $[-f_s/2, f_s/2]$. Thus, if the high-rate signal $x'(n')$ has frequency components outside the low-rate Nyquist interval $[-f_s/2, f_s/2]$, aliasing would occur and $x_{\text{down}}(n) \neq x(n)$.

Figure 1: Orfanidis, p. 687, Figure 12.5.1

In Fig. 12.5.1, the input spectrum was already restricted to the f_s Nyquist interval, and therefore, aliasing did not occur. The rate decrease causes the spectral images of $X'(f)$ at multiples of f'_s to be down-shifted and become images of $X(f)$ at multiples of f_s without overlapping. The mathematical justification of this down-shifting property is derived by expressing Eq. (12.5.2) in the frequency domain (see Problem 12.12)

$$X(f) = X_{\text{down}}(f) = \frac{1}{L}\sum_{m=0}^{L-1} X'(f - mf_s) \tag{2}$$

Therefore, the downsampling process causes the periodic replication of the original spectrum $X'(f)$ at multiples of the low rate f_s. This operation is depicted in Fig. 12.5.2 for $L = 4$.

To avoid the aliasing that will arise by the spectrum replication property (12.5.3), the high-rate input $x'(n')$ must be prefiltered by a digital lowpass filter, called the *decimation* filter. The combined filter/downsampler system is called a decimator and is depicted in Fig. 12.5.3.

Figure 2: Orfanidis, p. 688, Figure 12.5.2

Figure 3: Orfanidis, p. 688, Figure 12.5.3

The filter operates at the high rate f_s' and has cutoff frequency $f_c = f_s/2 = f_s'/(2L)$. It is similar to the ideal interpolation filter, except its DC gain is unity instead of L. The high-rate output of the filter is downsampled to obtain the desired low-rate decimated signal, with non-overlapping down-shifted replicas:

$$y_{\text{down}}(n) = y'(nL) \quad \leftrightarrow \quad \frac{1}{L}\sum_{m=0}^{L-1} Y'(f - mf_s) \tag{3}$$

Because only every Lth output of the filter is needed, the overall computational rate can be reduced by a factor of L using polyphase filters.

3 Sampling Rate Converters (Section 12.6)

Interpolators and decimators are examples of sampling rate converters that change the rate by integer factors. A more general sampling rate converter can change the rate by an arbitrary rational factor, say L/M, so that the output rate will be related to the input rate by

$$f_s' \;=\; \frac{L}{M} f_s \tag{4}$$

Such rate changes are necessary in practice for interfacing DSP systems operating at different rates. For example, to convert digital audio at the standard rate of 48 kHz to CD audio at 44.1 kHz, one must use the factor $44.1/48 = 147/160$.

The rate conversion can be accomplished by first increasing the rate by a factor of L to the high rate $f_s'' = Lf_s$ using an L-fold interpolator, and then decreasing the rate by a factor of M down to $f_s' = f_s''/M = Lf_s/M$ using an M-fold decimator.

Note that f_s'' is an integer multiple of both the input and output rates

$$f_s'' = Lf_s = Mf_s' \tag{5}$$

Because both the interpolation and decimation filters are operating at the same high rate f_s'' and both are lowpass filters, they may be combined into a single lowpass filter preceded by an upsampler and followed by a downsampler, as shown in Fig. 12.6.1. The interpolation filter must have cutoff frequency $f_s''/(2L) = f_s/2$ and the decimation filter $f_s''/(2M) = f_s'/2$. Thus, the cutoff frequency of the common filter must be chosen to be the *minimum* of the two

$$f_c = \frac{1}{2}\min(f_s, f_s') \tag{6}$$

Figure 4: Orfanidis, p. 692, Figure 12.6.1

When $f_s' > f_s$, the common filter acts as an anti-image postfilter for the upsampler, removing the spectral replicas at multiples of f_s but not at multiples of Lf_s. When $f_s' < f_s$, it acts as an antialiasing prefilter for the downsampler, making sure that the down-shifted replicas at multiples of f_s' do not overlap.