1 Lecture Outline

Reading: Chapter 1 Sampling and Reconstruction

- Introduction to sampling
- Ideal sampling
- Non-ideal sampling
- Spectrum replication

2 Introduction to Sampling

Under *certain* conditions, an analog or continuous-time (CT) signal can be completely represented by values at equally-spaced points in time. These values are known as *samples* and the resulting sampled signal is discrete-time (DT). The "completely represented" means that the CT signal can be uniquely specified by the samples or that the signal's frequencies or spectrum can be preserved under sampling.

Figure 1: Illustration of sampling and reconstruction

In addition, under the same conditions as above, the DT signal can be used to *reconstruct* the original CT signal even though the signal's values between sample points has been lost. The Nyquist *sampling theorem* provides a basis for signal sampling and reconstruction and is of profound importance in Electrical Engineering and many other fields.

Much of the importance of the sampling theorem also lies in its role as a bridge between CT and DT signals. The fact that under certain conditions a CT signal can be completely recovered from a sequence of its samples provides a mechanism for representing a CT signal by a DT signal. In addition, as compared to equivalent CT signal processing systems, DT systems are often lower cost, require lower power, are physically smaller and lighter, and are more flexible and reprogrammable.

3 Representation of a CT Signal by its Samples

In general, we would not expect that a CT signal could be uniquely specified by a sequence of of equally-spaced samples. Clearly an infinite number of signals can generate a given set of samples.

As we will see, however, if a signal is *band-limited* (signal spectrum is zero above a certain frequency) and if the samples are made fast enough (sampling frequency or sample rate), then the samples uniquely specify the signal and we can reconstruct it perfectly.

Figure 2: CT and equivalent DT signal processing systems

Figure 3: Ambiguity in the DT representation

4 Ideal Sampling

In order to develop the sampling theorem, we must mathematically model the sampling of a CT signal at regular intervals. A convenient way to do this is to multiply the CT signal we wish to sample with a periodic impulse train. This modeling is known as *ideal sampling* or *impulse-train sampling*.

Figure 4: Orfanidis Figure 1.5.1 (ideal sampling)

Let $x(t)$ denote our CT signal. We denote the impulse-train with period T or the *sampling function* as

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \tag{1}$$

and the "sampled signal" as

$$\hat{x}(t) = x(t)s(t) \tag{2}$$

or

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT). \tag{3}$$

Note that as used above, T is the sampling period, $\Omega_s = 2\pi/T$ is the sampling frequency (rads/s), and $f_s = 1/T$ is the sampling frequency (samples/s).

5 Non-Ideal Sampling

In *non-ideal sampling* or *practical sampling*, each sample must be held constant for a short period of time, say τ seconds, in order for the A/D converter to accurately convert the sample to digital format. This holding operation may be achieved by a sample/hold circuit. In this case, the sampled signal will be:

$$x_{\text{flat}}(t) = \sum_{n=-\infty}^{\infty} x(nT)p(t - nT). \tag{4}$$

where $p(t)$ is a flat-top pulse of duration of τ seconds such that $\tau \ll T$. Ideal sampling corresponds to the limit $\tau \to 0$. Figure 1.5.1 illustrates the practical case.

Figure 5: Orfanidis Figure 1.5.1 (non-ideal sampling)

6 Spectrum Replication

The spectrum of the sampling function is calculated as follows. The Fourier Series of $s(t)$ is given by

$$s(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_s t} \tag{5}$$

where

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} s(t)e^{-jk\Omega_s t} \\ &= 1/T. \end{aligned} \tag{6}$$

Thus the Fourier Transform of (5), yields the spectrum of the sampling function

$$\begin{aligned} S(j\Omega) &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_s) \\ &= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s). \end{aligned} \tag{7}$$

Furthermore, we know that multiplication in the time-domain is equivalent to convolution in the frequency-domain. Therefore the spectrum of the sampled signal, $\hat{X}(j\Omega)$ is the frequency-domain version of (3)

$$\begin{aligned}
\hat{X}(j\Omega) &= X(j\Omega) * S(j\Omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)S(j(\Omega-\theta))d\theta \\
&= \frac{1}{T}\int_{-\infty}^{+\infty} X(j\theta)\sum_{k=-\infty}^{\infty}\delta(\Omega-\theta-k\Omega_s)d\theta \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{+\infty} X(j\theta)\delta(\Omega-\theta-k\Omega_s)d\theta \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega-k\Omega_s)) \qquad (8)
\end{aligned}$$

We interpret (8) as follows. The spectrum of the sampled signal, $\hat{X}(j\Omega)$ is a periodic function of Ω consisting of a superposition of shifted replicas of the original signal spectrum $X(j\Omega)$. The scaling is by $1/T$ and the shifting by multiples Ω_s.

Figure 6: Orfanidis Figure 1.5.2

Note that if $f_{\max} < (f_s - f_{\max})$ or equivalently $f_s > 2f_{\max}$ there is no overlap between the replicas of $X(j\Omega)$. When there is no overlap, the original signal spectrum is preserved and there is no spectral distortion. On the other hand, when the spectral replicas overlap and add (superposition), there is spectral distortion. This spectral distortion is known as *aliasing* and cannot be undone. Preservation of the original signal spectrum suggests that the original CT signal can be faithfully reproduced by lowpass filtering $\hat{X}(j\Omega)$ (to filter out the replicas).

Figure 7: Orfanidis Figure 1.5.3

Nyquist Sampling Theorem (1928): Let $x(t)$ be a bandlimited signal with $X(j\Omega) = 0$ for $|\Omega| > \Omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$ if

$$f_s > 2f_{\max}. \qquad (9)$$

where f_s is in units of samples/s and the highest frequency in $x(t)$ is given by f_M in cycles/s or Hz. The frequency $f_{\max}$ is referred to as the Nyuist frequency and rate $2f_{\max}$ is referred to as the *Nyquist rate*.