1 Lecture Outline

Reading: Chapter 11 IIR Digital Filter Design

This lecture will cover the following topics

- Bilinear Transformation (Section 11.1)
- Procedure for Designing Digital Filters using the BLT
- Analog Butterworth Lowpass Filter (Section 11.6.1)
- Digital Butterworth Lowpass Filter (Section 11.6.2)

2 Bilinear Transformation

Instead of designing the digital filter directly from the digital filter specifications, we will first convert to equivalent analog filter specifications. Then these analog specifications are used to design an analog filter $H_a(s)$ using many, widely available analog filter design techniques. Finally, the *bilinear transformation* (BLT) method maps the analog filter $H_a(s)$ to the equivalent, desired digital filter, $H(z)$.

Figure 1: Orfanidis, p. 563, Figure 11.1.1

The mapping to and from continuous time/analog domain to discrete time/digital domain is performed with an algebraic transformation between the variables s and z that maps the entire $j\Omega$-axis in the s-plane to one revolution of the unit circle, $e^{j\omega}$ in the z-plane.

The bilinear transformation is a particular map between the variables s and z. For example, to map an analog LPF prototype in the s-plane to a digital LPF in the z-plane we use[1]

$$s = \frac{1-z^{-1}}{1+z^{-1}}. \tag{1}$$

A useful property of the BLT is that it maps the left-hand s-plane into the inside of the unit circle on the z-plane. Figure 11.1.2 shows this property. Because all analog filter design methods give rise to stable and causal transfer functions $H_a(s)$, this property guarantees that the digital filter $H(z)$ will also be stable and causal.

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[1] Other transformations can map an analog LPF prototype a digital HPF, digital BPF, etc.

Figure 2: Orfanidis, p. 565, Figure 11.1.2

The corresponding map of frequencies is obtained by replacing s with $j\Omega$ and z with $e^{j\omega}$ in the BLT:

$$\begin{aligned} j\Omega &= \frac{1-e^{-j\omega}}{1+e^{-j\omega}} \qquad (2) \\ &= \frac{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}+e^{-j\omega/2}\right)} \\ &= j\frac{2\sin(\omega/2)}{2\cos(\omega/2)} \\ &= j\tan(\omega/2) \end{aligned}$$

Therefore, in the frequency mapping, we *warp* the frequencies according to

$$\Omega = \tan(\omega/2) \qquad (3)$$

or

$$\omega = 2\arctan(\Omega). \qquad (4)$$

Figure 3: Frequency warping and Orfanidis, p. 567, Figure 11.2.2

Procedure for designing a digital LPF using the Bilinear Transformation:

Given magnitude response specifications for digital LPF

1. Prewarp digital filter specifications using frequency map, $\Omega = \tan(\omega/2)$
2. Design an equivalent analog LPF $H_a(s)$ to meet the analog specifications in Step 1
3. Apply bilinear transformation $s = \frac{1-z^{-1}}{1+z^{-1}}$ to $H_a(s)$ to obtain desired $H(z)$

Example. Suppose we wish to design a digital LPF with cutoff frequency $\omega_c = \pi/2$ (rads/sample). We would do the following.

1. Compute $\Omega_c = \tan(\omega_c/2) = 1$ (rads/s)

2. Design an analog LPF $H_a(s)$, with cutoff frequency of $\Omega_c = 1$ (rads/s)
3. Compute the digital LPF as

$$\begin{aligned} H(z) &= \left. H_a(s) \right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \\ &= H_a\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \end{aligned} \tag{5}$$

3 Analog Butterworth Lowpass Filter (Section 11.6.1)

The above procedure requires the design of an analog LPF $H_a(s)$. One popular analog filter design is known as the Butterworth and has a magnitude-squared response defined by

$$|H_a(s)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \tag{6}$$

where the two design parameters are filter order, N and 3 dB down frequency Ω_c. We observe the following:

- Magnitude response is monotonic in passband and stopband
- As N increases filter characteristics become sharper (they remain closer to unity over more of the passband and become close to zero more rapidly in the stopband)
- At the cutoff frequency, $|H_a(\Omega_c)|^2 = 1/2 = 3$ dB
- Maximally flat passband at DC (first $2N-1$ derivatives at $\Omega = 0$ are zero)

The order N analog Butterworth lowpass filter with cutoff frequency Ω_c is written in factored form as

$$\begin{aligned} H_a(s) &= \prod_{i=1}^{N/2} H_{a,i}(s), \quad N \text{ even} \\ H_a(s) &= H_{a,0}(s) \prod_{i=1}^{(N-1)/2} H_{a,i}(s), \quad N \text{ odd} \end{aligned} \tag{7}$$

where the factors are given by

$$H_{a,i}(s) = \frac{1}{1 - 2\frac{s}{\Omega_c}\cos(\theta_i) + \frac{s^2}{\Omega_c^2}}, \tag{8}$$

$$H_{a,0}(s) = \frac{1}{1+\frac{s}{\Omega_c}}, \tag{9}$$

and

$$\theta_i = \frac{\pi}{2N}(N-1+2i), \quad 1 \le i \le N. \tag{10}$$

Analog Butterworth LPF transfer functions $H_a(s) = 1/D(s)$ (in factored form) are given p. 599 Table 11.6.1.

4 Digital Butterworth Lowpass Filter (Section 11.6.2)

To derive the order N digital Butterworth lowpass filter $H(z)$ we apply the BLT to the order N analog Butterworth lowpass filter $H_a(s)$ with cutoff frequency Ω_c. To simplify the transformation, we work with each second order factor:

$$\begin{aligned} H_i(z) &= \left. H_{a,i}(s)\right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \\ &= \left. \frac{1}{\left(1 - 2\frac{s}{\Omega_c}\cos\theta_i + \frac{s^2}{\Omega_c^2}\right)} \right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \\ &= \frac{1}{1 - \frac{2}{\Omega_c}\cos\theta_i\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \frac{1}{\Omega_c^2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2} \\ &= \frac{b'_{0,i} + b'_{1,i}z^{-1} + b'_{2,i}z^{-2}}{1 + a'_{1,i}z^{-1} + a'_{2,i}z^{-2}} \end{aligned} \tag{11}$$

where

$$b'_{0,i} = b'_{2,i} = \frac{\Omega_c^2}{1 - 2\Omega_c\cos\theta_i + \Omega_c^2}, \tag{12}$$

$$b'_{1,i} = 2b'_{0,i}, \tag{13}$$

$$a'_1 = \frac{2\left(\Omega_c^2 - 1\right)}{1 - 2\Omega_c\cos\theta_i + \Omega_c^2}, \tag{14}$$

$$a'_2 = \frac{1 + 2\Omega_c\cos\theta_i + \Omega_c^2}{1 - 2\Omega_c\cos\theta_i + \Omega_c^2} \tag{15}$$

For the odd filter order case, the first order factor becomes

$$\begin{aligned} H_0(z) &= \left. H_{a,0}(s)\right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \\ &= \left. \frac{1}{\left(1 + \frac{s}{\Omega_c}\right)} \right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \\ &= \frac{1}{1 + \frac{1}{\Omega_c}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{b'_{0,0} + b'_{1,0}z^{-1}}{1 + a'_{1,0}z^{-1}} \end{aligned} \tag{16}$$

where

$$b'_{0,0} = b'_{1,0} = \frac{\Omega_c}{\Omega_c + 1}, \tag{17}$$

$$a'_{1,0} = \frac{\Omega_c - 1}{\Omega_c + 1}. \tag{18}$$

Finally, the overall transfer function is the product of the above factors

$$\begin{aligned} H(z) &= \prod_{i=1}^{N/2} H_i(z), \quad N \text{ even} \\ H(z) &= H_0(z) \prod_{i=1}^{(N-1)/2} H_i(z), \quad N \text{ odd} \end{aligned} \tag{19}$$

In the EE395 DSP Toolkit (and in MATLAB's Signal Processing Toolbox), there is an implementation of the above digital Butterworth LPF design formulas. Its use is as follows.

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[b,a] = bttrwrth(N,wc);
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