1 Lecture Outline

Reading: Chapter 10 FIR Digital Filter Design

This lecture will cover the following topics

- Introduction
- (Rectangular) Window Method for FIR Filters
- Designing for Linear Phase Response
- Hamming Window for FIR Filters
- Shifting and Modulating LPFs

2 Introduction

We would like to design a digital filter that is functionally equivalent to an analog filter:

Figure 1: Ludeman, p.168, Fig. 4.1

There are three steps to the design of a filter:

1. the specification of the frequency response of the system
2. the approximation of the specifications using a causal, DT system
3. the realization of the system (FIR coefficient vector **h** or IIR coefficient vectors **b** and **a**).

Table 1: FIR vs. IIR Filter

FIR (Advantages)	IIR (Advantages)
possibility of linear phase response	low computational cost
guaranteed stability	efficient implementation (Cascade of SOS)

Table 2: FIR vs. IIR Filter

FIR (Disadvantages)	IIR (Disadvantages)
large N for sharp filters which implies high computational cost	potential instability due to coefficient quantization
	never linear phase over full Nyquist interval

The most popular filter design techniques (and associated MATLAB tools) are:

FIR Windowed sinc functions (fir1.m) and optimum approximations (remez.m)

IIR Bilinear Transformation (butterworth and chebyshev)

3 (Rectangular) Window Method for FIR Filters

3.1 Ideal Filters

The window method is one of the simplest methods for FIR filter design and is well suited for filters with simple frequency responses such as the ideal low pass filter (LPF). In this case the desired frequency response

Figure 2: Orfanidis p. 533 Figure 10.1.1 (LPF).

is defined as

$$D(\omega) = \begin{cases} 1, & -\omega_c \le \omega \le \omega_c \\ 0, & -\pi \le \omega < -\omega_c \text{ or } \omega_c < \omega \le \pi \end{cases} \tag{1}$$

The impulse response can be obtained by IDTFT

$$\begin{aligned} d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{j2\pi n} \\ &= \frac{\sin(\omega_c n)}{\pi n} \\ &= \frac{\omega_c}{\pi}\mathrm{sinc}(\omega_c n), \quad -\infty < n < \infty \end{aligned} \tag{2}$$

Obviously, the impulse response of the ideal LPF is more an academic than a practical result:

1. not computationally realizable due to infinitely long impulse response whose transfer function is not rational
2. double-sided and is non-causal. We will see shortly that it can be used as a starting point for practical designs.
3. not stable since it is not square-summable

3.2 Rectangular Window

The first approximation to the ideal filter will be to truncate (rectangularly window) its double-sided infinite impulse response to a finite length (FIR)

$$\mathbf{d} \quad = \quad [d(-M) \ \ \ldots \ \ d(-1) \ \ d(0) \ \ d(1) \ \ \ldots \ \ d(M)]^T \tag{3}$$

where

$$d(n) \quad = \quad \frac{\omega_c}{\pi}\mathrm{sinc}(\omega_c n). \tag{4}$$

Here the length of $\mathbf{d}$ is $N = 2M + 1$ and it is still non-causal. We need only shift $\mathbf{d}$ by M samples to have a causal filter (assume N is odd and $M = (N - 1)/2$)

$$h(n) \quad = \quad d(n - M), \quad 0 \le n \le N - 1. \tag{5}$$

Since $\mathbf{h}$, $\mathbf{d}$ are approximations to the ideal filter we expect the frequency response to be an approximation to the ideal.

Steps for length-N FIR, rectangular-window method, LPF (cutoff ω_c):

1. Pick an odd length $N = 2M + 1$ and let $M = (N - 1)/2$
2. Calculate the N coefficients $d(n) = \frac{\omega_c}{\pi}\mathrm{sinc}(\omega_c n)$, $-M \le n \le M$
3. Filter impulse response $h(n) = d(n - M)$, $\ 0 \le n \le N - 1$

Figure 3: Orfanidis p. 539 Figure 10.1.4 (LPF).

The FIR approximation to the ideal LPF can be thought of as a rectangular-windowed version of the ideal impulse response

$$h(n) \quad = \quad w(n)d(n - M), \quad \infty \le n \le \infty. \tag{6}$$

where $w(n)$ is the length-N rectangular window and $N = 2M + 1$. As we saw earlier when studying spectral analysis, this windowing can be viewed in the frequency domain as

$$H(\omega) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega - \xi)e^{-j\xi M}D(\xi)d\xi \tag{7}$$

where $e^{-j\xi M}$ arises from the M-sample shift required for causality. This windowing convolves or "smears" the ideal response $D(\omega)$ and the window $W(\omega)$ to result in the frequency response

$$H(\omega) \quad = \quad \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} W(\omega - \xi)e^{-j\xi M}d\xi \tag{8}$$

As length of the FIR approximation N increases we observe three effects (p. 539 Orphanidis):

Figure 4: Orfanidis p. 539 Figure 10.1.5.

1. For ω's that lie well within the passband or stopband, the ripple size decreases with increasing N. This results in flatter (closer to the ideal) passband and stopband.
2. The transition width (bandwidth between passband and stopband) decreases with increasing N. For any N, $|H(\omega_c)| = 1/2$ (half-amplitude). Note that many filter designs yield $|H(\omega_c)|^2 = 1/2$ (half-power)
3. The largest ripples tend to cluster near the passband-to-stopband discontinuity and are independent with N. Behavior tied to Gibbs phenomenon in Fourier Series.

4 Designing for Linear Phase Response

We can show window designs have a *linear phase response*. Let **h** be a length-N FIR filter with system function given by $H(z)$. The phase term $\angle H(\omega)$ was defined as

$$H(\omega) \quad = \quad |H(\omega)|e^{j\angle H(\omega)}. \tag{9}$$

Therefore $H(z)$ is said to have linear phase if

$$H(\omega) \quad = \quad A(\omega)e^{j(\alpha\omega-\beta)}. \tag{10}$$

where $A(\omega)$ is a real and even function of ω and $\angle H(\omega)$ is linear. The magnitude response of H is the absolute value of A. The phase of H has the form

$$\angle H(\omega) \quad = \quad -\alpha\omega + \beta. \tag{11}$$

The term $-\alpha\omega$ corresponds to a delay of α samples (we studied this when we examined delays and phase delays). There are four ways to achieve the linear phase property with FIR filters depending on the length of **h** as well as its symmetry.

Type I N is even; $h(n) = h(N-n)$, $0 \le n \le N$ (even symmetry)

Type II N is odd; $h(n) = h(N-n)$, $0 \le n \le N$ (even symmetry)

Type III N is even; $h(n) = -h(N-n)$, $0 \le n \le N$ (odd symmetry)

Type IN N is odd; $h(n) = -h(N-n)$, $0 \le n \le N$ (odd symmetry)

5 Hamming Window for FIR Filters

To reduce the passband and stopband ripples caused by the large sidelobes of the rectangular window frequency response, we consider choosing an alternate window which tapers off at its endpoints thus reducing the large sidelobes and subsequently reduce the passband and stopband ripples. Of course the price paid for choosing a non-rectangular window is a wider mainlobe which implies that the resulting filter will have a wider transition band.

Steps for Hamming window method of length-N FIR LPF design

1. Pick an odd length $N = 2M + 1$ and let $M = (N - 1)/2$
2. Calculate the N coefficients $d(n) = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$.
3. Filter impulse response $h(n) = w(n)d(n - M), \quad 0 \leq n \leq N - 1$ where

$$w(n) \quad = \quad 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \leq n \leq N-1 \tag{12}$$

Figure 5: Orfanidis p. 540 Figure 10.1.6 and 10.1.7.

Table 3: Rectangular vs. Hamming Window FIR Filter Designs

Rectangular Window Designs	Hamming Window Designs
narrowest transition band	wider transition band
poor stopband attenuation (∼21dB)	better stopband attenuation (∼54dB)
8.9% passband/stopband ripple (overshoot)	0.2% passband/stopband ripple (overshoot)

6 Shifting and Modulating LPFs for Other Response Types

The above designs window the impulse response of an LPF. Of course, the same windowing techniques can be applied to impulse responses for other filter types such as high pass filters (HPFs) and band pass filters (BPFs). Alternatively, we can design a prototype LPF and frequency shift or modulate the response to a desired center frequency. Such a frequency transformation enables us to design other filter types from the LPF prototype.

6.1 Frequency Shifting

The frequency shift property of the DTFT is given by

$$g(n) = h(n)e^{j\omega_c n} \quad \leftrightarrow \quad G(\omega) = H(\omega - \omega_c) \tag{13}$$

where $h(n)$ is the impulse response of the LPF prototype and ω_c is the frequency to shift to. The result is shown below. Note that if $g(n)$ is a real-valued impulse response, $h(n)$ is complex-valued (except for $\omega_c = \pi$) and thus has a single-sided frequency response.

Figure 6: Frequency-shifting a LPF prototype

6.2 Modulating

The modulation property of the DTFT is given by

$$g(n) = h(n)\cos(\omega_c n) \quad \leftrightarrow \quad G(\omega) = \frac{1}{2}\left[H(\omega - \omega_c) + H(\omega + \omega_c)\right] \tag{14}$$

where $h(n)$ is the impulse response of the LPF prototype and ω_c is the frequency to shift to. The result is shown below. Note that if $g(n)$ is a real-valued impulse response, $h(n)$ is real-valued.

Figure 7: Modulating a LPF prototype