1 Lecture Outline

Reading: Chapter 9 DFT/FFT Algorithms

- Inverse DFT (Section 9.6)

2 Inverse DFT (Optional)

Inversion of the N-point DFT involves finding the length L signal x such that

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{1}$$

i.e. analysis into N Fourier components, where $\mathbf{W}$ is the $N \times L$ DFT matrix

$$\mathbf{W} = \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^{L-1} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{N-1} & \cdots & W_N^{(N-1)(L-1)} \end{bmatrix} \tag{2}$$

2.1 Matrix Analysis Notes

When:

- $L < N$ ($\mathbf{W}$ is not invertible), our system of equations is *overdetermined*, i.e. more equations than unknowns and *no solution exists.*
- $L = N$ ($\mathbf{W}$ is invertible), a *unique solution exists.*
- $L > N$ ($\mathbf{W}$ is not invertible), our system of equations is underdetermined, i.e. more unknowns than equations and an *infinite number of solutions exist.*

2.2 Inverse DFT Solutions

For $L = N$, the inverse DFT is given by

$$\mathbf{x} = \mathbf{W}^{-1}\mathbf{X} \tag{3}$$

For $L > N$, the inverse DFT has many possible solutions for $\mathbf{x}$–all of which have the same $\tilde{\mathbf{x}}$. Among the solutions, the only one uniquely obtainable from $\mathbf{X}$ is $\tilde{\mathbf{x}}$ (since $\mathbf{W}$ is not invertible):

$$\begin{aligned} \mathbf{X} &= \mathbf{W}\mathbf{x} \\ &= \tilde{\mathbf{W}}\tilde{\mathbf{x}} \end{aligned} \tag{4}$$

Thus the inverse DFT is therefore always defined as (synthesis from N Fourier components)

$$\tilde{\mathbf{x}} = \tilde{\mathbf{W}}^{-1}\tilde{\mathbf{X}}. \tag{5}$$

Note: If we begin with an $L \times 1$ vector of samples, $\mathbf{x}$ and compute the N-point DFT as

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{6}$$

then when we compute the inverse, we'll end up with a vector of samples (assuming $L \neq N$) which is different than what we started with

$$\tilde{\mathbf{x}} = \tilde{\mathbf{W}}^{-1}\tilde{\mathbf{X}}. \tag{7}$$

3 Unitary DFT Matrix

The elements of $\tilde{\mathbf{W}}^{-1}$ may be obtained without first having to invert $\tilde{\mathbf{W}}$. The DFT matrix is *unitary* (^{H} is matrix complex conjugate transpose) thus,

$$\frac{1}{N}\tilde{\mathbf{W}}\tilde{\mathbf{W}}^H = \mathbf{I} \tag{8}$$

and so

$$\tilde{\mathbf{W}}^{-1} = \frac{1}{N}\tilde{\mathbf{W}}^H \tag{9}$$

Therefore we have

$$\begin{aligned} \tilde{\mathbf{x}} &= \tilde{\mathbf{W}}^{-1}\mathbf{X} \\ &= \frac{1}{N}\tilde{\mathbf{W}}^*\mathbf{X} \end{aligned} \tag{10}$$

We see that the IDFT (synthesis equation) is identical to the DFT (analysis equation) except for the $1/N$ scale factor and inverse powers of $\mathbf{W}^{kn}$. Therefore, it is reasonable to expect an inverse DFT (IDFT) can be obtained from the DFT with some minor adjustments

$$\begin{aligned} \tilde{\mathbf{x}} &= \tilde{\mathbf{W}}^{-1}\mathbf{X} \\ &= \frac{1}{N}\tilde{\mathbf{W}}^H\mathbf{X} \\ &= \frac{1}{N}\left(\tilde{\mathbf{W}}\mathbf{X}^*\right)^*, \quad \text{since } \mathbf{W} \text{ is symmetric} \\ &= \frac{1}{N}\left\{\mathrm{DFT}\left[\mathbf{X}^*\right]\right\}^* \end{aligned} \tag{11}$$

The $1/N$ scaling can be achieved by a per stage scaling of $1/2$ when N is a power of two.

