1 Lecture Outline

Reading: Chapter 9 DFT/FFT Algorithms

- Matrix Form of the DFT (Section 9.4)
- Modulo-N Reduction (Section 9.5)

2 Matrix Form of the DFT

Consider the N-point DFT of a length L signal

$$X(k) = \sum_{n=0}^{L-1} x(n) e^{-j\omega_k n} \tag{1}$$

where

$$\omega_k = 2\pi k/N, \quad 0 \le k \le N-1 \tag{2}$$

is the kth DFT frequency. Substitution yields

$$X(\omega_k) = \sum_{n=0}^{L-1} x(n) e^{-j2\pi kn/N}. \tag{3}$$

For convenience, let the Nth root of unity (often called a twiddle factor) be denoted as

$$W_N = e^{-j2\pi/N}. \tag{4}$$

Example:

$$W_2 = e^{-j2\pi/2} = -1 \tag{5}$$

$$W_4 = e^{-j2\pi/4} = -j \tag{6}$$

$$W_8 = e^{-j2\pi/8} = (1-j)/\sqrt{2} \tag{7}$$

The DFT can now be rewritten as (replacing ω_k with the simple index k)

$$X(k) = \sum_{n=0}^{L-1} x(n) W_N^{kn}, \quad 0 \le k \le N-1. \tag{8}$$

We can consolidate the above N equations in matrix form

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^{(0)(0)} & W_N^{(0)(1)} & \cdots & W_N^{(0)(L-1)} \\ W_N^{(1)(0)} & W_N^{(1)(1)} & \cdots & W_N^{(1)(L-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1)(0)} & W_N^{(N-1)(1)} & \cdots & W_N^{(N-1)(L-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(L-1) \end{bmatrix} \tag{9}$$

or

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{10}$$

The N-point DFT can be thought of as a linear matrix (unitary) transformation (using W, the DFT matrix) of the L-dimensional vector of time data into an N-dimensional vector of frequency samples.

Figure 1: Fourier transforms as a change of basis

3 Modulo-N Reduction

Modulo-N reduction or wrapping is achieved by segmenting the length-L signal $\mathbf{x}$ into contiguous non-overlapping blocks $\mathbf{x}_i$ of length N, wrapping the blocks to be time-aligned with the first block, and adding them up to form $\tilde{\mathbf{x}}$. Consider this operation as a time-domain aliasing.

Example: Determine the mod-3 reduction of the length-8 signal vector

$$\mathbf{x} = [1 \;\; 2 \;\; -2 \;\; 3 \;\; 4 \;\; -2 \;\; -1 \;\; 1]^T \tag{11}$$

We divide $\mathbf{x}$ into length-3 blocks and sum (overlap and add)

$$\begin{aligned} \tilde{\mathbf{x}} &= \mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2 \\ &= \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} \end{aligned} \tag{12}$$

The connection of the mod-N reduction to the DFT is that the length-N wrapped signal, $\tilde{\mathbf{x}}$ has the *same* N-point DFT as the original, length-L unwrapped signal, $\mathbf{x}$

$$\tilde{\mathbf{X}}(k) = \mathbf{X}(k) \tag{13}$$

where

$$\tilde{\mathbf{X}}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn}, \quad 0 \le k \le N-1. \tag{14}$$

4 Final Comments

We can mathematically prove that the N-point DFT of the unwrapped signal is the same as the N-point DFT of the modulo-N wrapped signal (see proof in text).

Why is this useful? We've shown that the N-point DFT of an L-length sequence takes NL complex MACs (brute force algorithm). Furthermore, we've shown that this DFT is equivalent to an N-point DFT on the modulo-N wrapped sequence which takes N^2 complex MACs. Performing the modulo-N wrapping requires $N(M-1)$ additions where $L = MN$. Comparing we see

N-point DFT on length L sequence: $NL = MN^2 = N(NM)$

N-point DFT on modulo-N wrapped sequence: $N^2 + NM = N(N+M)$

Combining modulo-N wrapping with the FFT to perform the N-point DFT on length L sequence we have

N-point FFT on modulo-N wrapped sequence: $N \log_2(N)/2 + NM$.

We've seen that $\mathbf{x}$ and $\tilde{\mathbf{x}}$ have the same DFT. Furthermore, we realize that any signal that has the same modulo-N reduction as $\mathbf{x}$ will have the same DFT as $\mathbf{x}$. The modulo-N wrapped signal $\tilde{\mathbf{x}}$ is unique in the sense that it is the shortest signal of length N, that has the same N-point DFT as the signal $\mathbf{x}$.