1 Lecture Outline

Reading: Chapter 9 DFT/FFT Algorithms

- Implications of windowing (Section 9.1)
- Other windows
- DTFT Computation (Section 9.2)
- DFT Computation

2 Implications of Windowing and "Smeared" Spectral Lines

If we have two complex exponentials (or sinusoids) at ω_1 and ω_2 which are separated in frequency such that the mainlobes do not overlap, we will have Fig. 9.1.3(b) and the complex exponentials are *resolvable*.

However, if $\Delta\omega = \omega_2 - \omega_1$ becomes small enough and the main lobes overlap, and the two complex exponentials do not appear distinct anymore and are no longer *resolvable*. Resolution of the two sinusoids requires that their frequency separation be greater than the main lobe width

$$\Delta\omega \quad \geq \quad 2\pi/L \tag{1}$$

The mainlobe width of $W(\omega)$ determines the amount of achievable frequency resolution.

The sidelobes determine the amount of *frequency leakage* and are undesirable artifacts of the windowing process. Sidelobes must be suppressed as much as possible because they may be confused with the mainlobes of weaker sinusoids that may be present. Note that the sidelobes are generated in the first place because of the abrupt off-on-off of the rectangular window.

3 Other Windows

The standard technique for suppressing sidelobes in the spectrum caused by the rectangular window is to multiply the signal (sample-by-sample) by a *non-rectangular* window. We'd like a window that tapers off to zero at the edges and therefore de-emphasizes the high frequencies introduced by the windowing process.

In general as we attenuate the sidelobes, we increase the mainlobe width thus reducing the frequency resolution capability of the windowed spectrum for fixed L.

There are many windows to choose from each with advantages and disadvantages. The classic windows paper (referenced many times in the literature) is

Harris, Fredric J., "On the use of windows for harmonic analysis with the discrete Fourier transform", *Proc. IEEE*, vol. 66, no. 1, Jan. 1978, p. 51–83.

One of the simplest and most widely used window is the *Hamming* window

$$w(n) \quad = \quad \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2\pi n}{L-1}\right), & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

The ratio of Hamming window's spectral mainlobe height to sidelobe height is at least 40 dB (rectangular is 13dB).

Figure 1: Orfanidis p. 471 9.1.4

For any window type (rectangular, Hamming, or other), the width of mainlobe is inversely proportional to window length (rectangular window, $c = 1$; Hamming window, $c = 2$)

$$\Delta\omega_W = c2\pi/L. \tag{3}$$

The minimum resolvable frequency difference between any two sinusoids will be

$$\Delta\omega \geq \Delta\omega_W = c2\pi/L. \tag{4}$$

Figure 2: Orfanidis p. 471 9.1.5–9.1.7

Therefore the minimum signal length required to achieve a given $\Delta\omega$ is c times longer than that for the rectangular window. Remember though that with the rectangular window, there is strong sidelobe presence.

Example 9.1.1: A signal consists of four sinusoids of frequencies 1, 1.5, 2.5, and 2.75 kHz and is sampled at 10 kHz. What is the minimum number of samples that should be collected for the frequency spectrum, to exhibit four distinct peaks at these frequencies? How many samples should be collected if we use a Hamming window?

Figure 3: Example 9.1.1

The smallest frequency separation that must be resolved by the DTFT is

$$\Delta f = 2.75 - 2.5 \text{ kHz} = 0.25 \text{ kHz} \tag{5}$$

or

$$\Delta\omega = 2\pi \cdot 0.25/10 = \pi/20 \text{ rads/sample}. \tag{6}$$

For a rectangular window we have

$$\Delta\omega \geq 2\pi/L \tag{7}$$

or

$$L \geq 40 \text{ samples.} \tag{8}$$

Because the mainlobe of a Hamming window is twice as wide as that of the rectangular window, it follows that twice as many samples must be collected, that is $L \geq 80$ samples.

4 DTFT Computation

4.1 DTFT at a Single Frequency

Consider a length L signal $x(n)$ $0 \leq n \leq L-1$ which may have been prewindowed by a length L non-rectangular window. The DTFT evaluated at a single frequency, ω_0 can be expressed as

$$X(\omega_0) = \sum_{n=0}^{L-1} x(n)e^{-j\omega_0 n} \tag{9}$$

Figure 4: Orfanidis p. 476 Figure 9.1.9

Using our MATLAB tools, this is implemented as

```
>> omega0 = 0.2;
>> X_omega0 = dtft(x,omega0)
```

4.2 DTFT over a Frequency Range

The DTFT expression can be computed over a range of frequencies

$$0 \leq \omega_a \leq \omega \leq \omega_b < 2\pi. \tag{10}$$

In either case the space between evaluation points (assuming N equally-spaced evaluation points) or "bin width" is

$$\Delta\omega_{\text{bin}} = (\omega_b - \omega_a)/N \tag{11}$$

Example: If we wish to compute 10 points in the spectrum over the range $0.25 \leq \omega < 0.35$ band using our dtft.m tool, we would have

Figure 5: Orfanidis p. 488 Figure 9.2.1

Figure 6: Orfanidis p. 480 Figure 9.2.2

```
N = 10;
bin_width = (0.35-0.25)/N;
w = [0.25:bin_width:0.35];
X = dtft(x,w);
```

4.3 DFT

The N-point DFT of a length L signal is defined to be the DTFT evaluated at N *equally-spaced* frequencies over the *full* Nyquist interval $0 \leq \omega < 2\pi$. The "DFT frequencies" are defined as

$$\omega_k = 2\pi k/N \tag{12}$$

i.e. uniformly spaced-around the unit circle. Therefore the N-point DFT of a length L signal will be

$$X(\omega_k) = \sum_{n=0}^{L-1} x(n)e^{-j\omega_k n}, \quad 0 \leq k \leq N-1 \tag{13}$$

For discrete spectra, we typically know N and thus write $X(k)$ instead of $X(\omega_k)$ since ω_k is easily computed and we know what frequency we are talking about.

Example: Suppose the length of the signal is L samples. Within the dtft.m tool, for each evaluation frequency, we perform L complex multiplication and additions (MACs). Furthermore, we loop through N frequencies to compute $X(\omega)$. Therefore DTFT computation requires a total of NL complex MACs. If $L = N$, the DTFT algorithm as implemented requires N^2 MACs.

4.4 Zero Padding

In principle L and N can be specified independently of each other: L is the number of samples in the signal and N is the number of frequencies at which we evaluate the DFT. It will be convenient for development of the FFT that we assume $L = N$.

If $L < N$ we can pad the signal with $N - L$ zeros to make it length N. Clearly this has no effect on the DFT since the addition summands are zero.

