1 Lecture Outline

Reading: Chapter 9 DFT/FFT Algorithms

- DTFT Review
- Introduction
- Frequency resolution and windowing (Section 9.1)
- Analysis of windowing on the DTFT

2 Discrete-Time Fourier Transform (DTFT)

The Fourier transform for discrete-time (digital) signals (DTFT) has the form

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}, \quad \text{(analysis)} \\ &\updownarrow \\ x(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega n}d\omega, \quad \text{(synthesis)} \end{aligned} \tag{1}$$

The resulting spectrum is a complex-valued, continuous function in ω. In general, closed-form expressions for $X(\omega)$ can be determined for a handful of simple signals. For other signals, we numerically *approximated* this function with our dtft.m tool by evaluating $X(\omega)$ for *particular* values of ω.

We now wish to develop a Fourier transform for discrete-time signals which is a complex-valued, *discrete* function in frequency (DFT). The discrete Fourier transform (DFT) and its efficient implementation, the fast Fourier transform (FFT) have three major uses in DSP:

1. (fast) numerical computation of the frequency spectrum of a signal
2. efficient implementation of convolution via the FFT
3. channel (error control) and source (compression) coding of waveforms, such as wideband audio (music), speech or images for efficient transmission and storage.

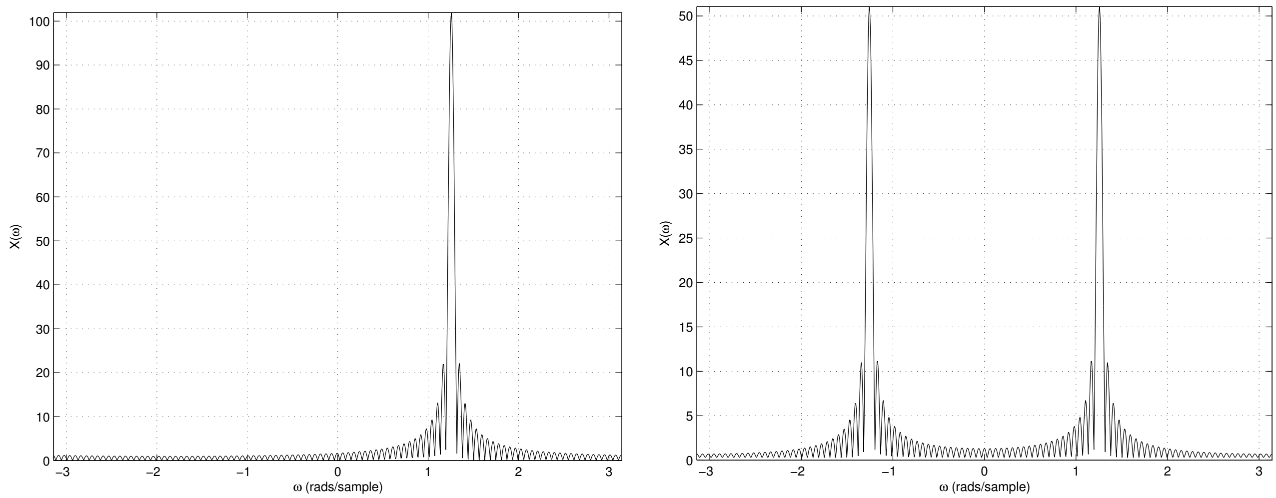
The DTFT as a function of frequency in Hz is given by

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi nf/f_s}, \quad \text{(analysis)} \tag{2}$$

where f_s is the sampling rate.

3 Introduction

Intuitively, since the complex-exponential basis functions in the DTFT are of infinite length, in order to achieve a good representation we should have an infinitely long signal. In practice, our signals will always be finite length. Therefore, we wish to answer two questions:



(a) (b)

Figure 1: DTFT of finite-length sinusoidal signals (a) complex exponential and (b) cosinusoid.

Q1: Does the spectrum of the finite-length version of a signal approach the spectrum of the infinite-length signal as the signal length goes to infinity?

Q2: What is the effect on the computed spectrum [approximation to $X(\omega)$] when we use a finite number of samples?

Recall the sinusoidal DTFTs:

$$x(n) = e^{j\omega_0 n} \quad \leftrightarrow \quad X(\omega) = 2\pi\delta(\omega - \omega_0) \tag{3}$$

$$x(n) = \cos(j\omega_0 n) \quad \leftrightarrow \quad X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \tag{4}$$

In Homework #4 we numerically computed these DTFTs based on *finite* length versions of the sinusoids. The resulting spectral were not sharp spectral lines but rather were *blurred* as shown in Fig. 1.

4 Frequency resolution and windowing

In order to answer the first question, we need to compare spectra of an infinitely long signal, $x(n)$ and its finite-length version $x_L(n)$ where L denotes the length of the signal. Therefore, we need to *model* the "shortening" of $x(n)$.

We can think of the finite length sequence $x_L(n)$ as a "rectangular windowed version" of the infinitely long sequence $x(n)$

$$x_L(n) \quad = \quad x(n)w_L(n) \tag{5}$$

where

$$w_L(n) \quad = \quad \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Figure 2: Orfanidis p. 465 Figure 9.1.1

We therefore express the spectrum of the finite-length signal, i.e. DTFT of the windowed signal as

$$\begin{aligned} X_L(\omega) &= \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x_L(n)e^{-j\omega n}. \end{aligned} \tag{7}$$

The DTFT of the windowed sequence is then computed for any desired value of ω.

Comparing (7) with (1), clearly as L increases, $x_L(n)$ becomes a better approximation to $x(n)$ and thus $X_L(\omega)$ becomes a better approximation to $X(\omega)$ thus establishing the answer to our first question.

5 Analysis of Windowing on the DTFT

We now examine the effect on the computed spectrum when we use a finite number of samples, i.e. windowing. In the time domain, our finite length signal model is

$$x_L(n) = x(n)w_L(n) \tag{8}$$

which implies that in the frequency domain we have convolution of the Fourier transforms

$$X_L(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\xi)W_L(\omega-\xi)d\xi \tag{9}$$

where $X(\omega)$ is the DTFT of $x(n)$ and $W_L(\omega)$ is the DTFT of $w_L(n)$. Thus the DTFT of the finite length signal is a convolved or “smeared” version of the DTFT of the infinite length signal with $W_L(\omega)$ thus answering our second question.

The frequency response of the “rectangular window” is given by

$$\begin{aligned} W(\omega) &= \sum_{n=-\infty}^{\infty} w_L(n)e^{-j\omega n} \\ &= \sum_{n=0}^{L-1} e^{-j\omega n} \\ &= \frac{1-e^{-jL\omega}}{1-e^{-j\omega}} \\ &= \frac{\sin(\omega L/2)}{\sin(\omega/2)}e^{-j\omega(L-1)/2} \end{aligned} \tag{10}$$

Features of the Rectangular Window:

Figure 3: Orfanidis p. 467 9.1.2: Magnitude response of rectangular window

1. main lobe of height L, main lobe width $2\pi/L$ (half base width), centered @ $\omega = 0$
2. side lobes @ $\omega = 2\pi k/L,\ k = \pm 1, \pm 2, \dots$ (1st sidelobe down 13 dB from main lobe)
3. as $L \to \infty$, main lobe height increases, main lobe width ($\Delta\omega_W = 2\pi/L$) decreases
4. as main lobe height increases, side lobe height also increase, however, ratio stays constant at -13 dB.

Example: Consider the complex exponential

$$\begin{aligned} x(n) &= e^{j\omega_1 n}, \ -\infty < n < \infty \\ &\updownarrow \\ X(\omega) &= 2\pi\delta(\omega - \omega_1). \end{aligned} \tag{11}$$

Now consider L samples of the complex exponential (windowed complex exponential)

$$x_L(n) = e^{j\omega_1 n},\ 0 \le n \le L-1. \tag{12}$$

The spectrum is

$$\begin{aligned} X_L(\omega) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\xi) W_L(\omega - \xi) d\xi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \xi) W_L(\omega - \xi) d\xi \\ &= W_L(\omega - \omega_1). \end{aligned} \tag{13}$$

Thus the windowing process smears the sharp spectral line of (11).

Figure 4: Orfanidis p. 469 9.1.3(a) and (b)