1 Lecture Outline

Reading: Chapter 1 Sampling and Reconstruction

- Sampling theorem (introduction)
- Antialising filters
- Analog reconstruction and aliasing
- Rotational motion
- DSP frequency units

2 Sampling Theorem

2.1 Introduction

Next, we study the sampling process, illustrated in Fig. 1.3.1, where the analog signal $x(t)$ is periodically measured every T seconds. Thus, time is discretized in units of the sampling interval T

$$t = nT, \quad n = 0, 1, 2, ... \tag{1}$$

Figure 1: Orfanidis Fig. 1.3.1

The sampling process represents a very drastic chopping operation on the original signal $x(t)$, and therefore, it will introduce a lot of spurious high-frequency components into the frequency spectrum—an effect known as *aliasing*. Thus, for system design purposes, two questions must be answered:

1. What is the effect of sampling on the original frequency spectrum?
2. How should one choose the sampling interval T?

Although the sampling process generates high frequency components, these components appear in a very regular fashion, that is, every frequency component of the original signal is periodically replicated over the entire frequency axis, with period given by the sampling rate

$$f_s = 1/T \tag{2}$$

2.2 Conditions for Proper Sampling

A more quantitative criterion is provided by the sampling theorem which states that for *accurate* representation of a signal $x(t)$ by its time samples $x(nT)$, two conditions must be met:

Figure 2: Orfanidis Fig. 1.3.2 Spectrum replication caused by sampling.

1. The signal $x(t)$ must be bandlimited, that is, its frequency spectrum must be limited to contain frequencies up to some maximum frequency, say $f_{\max}$, and no frequencies beyond that. A typical bandlimited spectrum is shown in Fig. 1.3.4.
2. The sampling rate f_s must be chosen to be at least twice the maximum frequency $f_{\max}$, that is,

$$f_s \geq 2f_{\max} \tag{3}$$

or, in terms of the sampling time interval: $T \leq \frac{1}{2f_{\max}}$.

Figure 3: Orfanidis Fig. 1.3.4 Typical bandlimited spectrum

2.3 Vocabulary

The minimum sampling rate allowed by the sampling theorem, that is, $f_s = 2f_{\max}$, is called the *Nyquist rate*.

For arbitrary values of f_s, the quantity $f_s/2$ is called the *Nyquist frequency* or *folding frequency*. It defines the endpoints of the *Nyquist frequency interval* $[-f_s/2, f_s/2]$.

The Nyquist frequency $f_s/2$ also defines the *cutoff frequencies* of the lowpass analog prefilters and postfilters that are required in DSP operations. The values of $f_{\max}$ and f_s depend on the application. Typical sampling rates for some common DSP applications are shown in Table on p. 7.

3 Antialiasing Filters (1.3.2)

The practical implications of the sampling theorem are quite important. Since most signals are not bandlimited, they must be made so by lowpass filtering *before* sampling. In order to sample a signal at a desired rate f_s and satisfy the conditions of the sampling theorem, the signal must be *prefiltered* by a lowpass *analog* filter, known as an *antialiasing* prefilter. The cutoff frequency of the prefilter, $f_{\max}$, must be taken to be at most equal to the Nyquist frequency $f_s/2$, that is, $f_{\max} \leq f_s/2$. This operation is shown in Fig. 1.3.5.

The output of the analog prefilter will then be *bandlimited* to maximum frequency $f_{\max}$ and may be sampled properly at the desired rate f_s. The spectrum replication caused by the sampling process can also be seen in Fig. 1.3.5.

It should be emphasized that the rate f_s must be chosen to be high enough so that, after the prefiltering operation, the surviving signal spectrum within the Nyquist interval $[-f_s/2, f_s/2]$ contains all the significant frequency components for the application at hand.

Figure 4: Orfanidis Fig. 1.3.5 Antialiasing prefilter

4 Analog Reconstruction and Aliasing (1.4.1)

Next, we discuss the aliasing effects that result if one violates the sampling theorem conditions (1.3.2). Consider the complex version of a sinusoid:

$$x(t) = e^{j\Omega t} = e^{j2\pi ft} \tag{4}$$

and its sampled version obtained by setting $t = nT$,

$$x(nT) = e^{j\Omega Tn} = e^{j2\pi fTn}. \tag{5}$$

Define also the following family of sinusoids, for $m = 0, \pm 1, \pm 2, ...$,

$$x_m(t) = e^{j2\pi(f+mf_s)t} \tag{6}$$

and their sampled versions,

$$x_m(nT) = e^{j2\pi(f+mf_s)Tn}. \tag{7}$$

Using the property $f_sT = 1$ and the trigonometric identity, $e^{j2\pi mfsTn} = e^{j2\pi mn} = 1$ we find that, although the signals $x_m(t)$ are different from each other, their sampled values are the same; indeed,

$$x_m(nT) = e^{j2\pi(f+mf_s)Tn} = e^{j2\pi fTn}e^{j2\pi mf_sTn} = e^{j2\pi fTn} = x(nT). \tag{8}$$

In terms of their sampled values, the signals $x_m(t)$ are indistinguishable, or aliased. Knowledge of the sample values $x(nT) = x_m(nT)$ is not enough to determine which among them was the original signal that was sampled. It could have been any one of the $x_m(t)$. In other words, the set of frequencies,

$$f, f \pm f_s, f \pm 2f_s, ..., f \pm mf_s, ... \tag{9}$$

are equivalent to each other. The effect of sampling was to replace the original frequency f with the replicated set. This is the intuitive explanation of the spectrum replication property depicted in Fig. 1.3.2. A more mathematical explanation will be given later using Fourier transforms.

Example 1.4.1: Consider a sinusoid of frequency $f = 10$ Hz sampled at a rate of $f_s = 12$ Hz. The sampled signal will contain all the replicated frequencies $10+m12$ Hz, $m = 0, \pm 1, \pm 2, ...$, or, $..., -26, -14, -2, 10, 22, 34, 46, ...$ and among these only $f_a = 10 \mod (12) = 10 - 12 = -2$ Hz lies within the Nyquist interval [-6, 6] Hz. This sinusoid will appear at the output of a reconstructor as a -2 Hz sinusoid instead of a 10 Hz one. On the other hand, had we sampled at a proper rate, that is, greater than $2f = 20$ Hz, say at $f_s = 22$ Hz, then no aliasing would result because the given frequency of 10 Hz already lies within the corresponding Nyquist interval of $[-11, 11]$ Hz.

Figure 5: Orfanidis p. 13 (five sinusoids)

5 Rotational Motion (1.4.2)

A more intuitive way to understand the sampling properties of sinusoids is to consider a representation of the complex sinusoid $x(t) = e^{j2\pi ft}$ as a wheel rotating with a frequency of f revolutions per second. The wheel is seen in a dark room by means of a strobe light flashing at a rate of f_s flashes per second. The rotational frequency in [radians/sec] is $\Omega = 2\pi f$. During the time interval T between flashes, the wheel turns by an angle:

$$\omega = \Omega T = 2\pi f T = \frac{2\pi f}{f_s}. \tag{10}$$

This quantity is called the *digital frequency* and is measured in units of [radians/sample]. It represents a convenient normalization of the physical frequency f. In terms of ω, the sampled sinusoid reads simply

$$x(nT) = e^{j2\pi fTn} = e^{j\omega n} \tag{11}$$

In units of ω, the Nyquist frequency $f = f_s/2$ becomes $\omega = \pi$ and the Nyquist interval becomes $[-\pi, \pi]$. The replicated set $f + mf_s$ becomes

$$\frac{2\pi(f + mf_s)}{f_s} = \frac{2\pi f}{f_s} + 2\pi m = \omega + 2\pi m \tag{12}$$

Because the frequency $f = f_s$ corresponds to $\omega = 2\pi$, the aliased frequency given in Eq. (1.4.3) becomes in units of ω:

$$\omega_a = \omega \mod (2\pi) \tag{13}$$

The quantity $f/f_s = fT$ is also called the digital frequency and is measured in units of [cycles/sample]. It represents another convenient normalization of the physical frequency axis, with the Nyquist interval corresponding to $[-0.5, 0.5]$. In terms of the rotating wheel, fT represents the number of revolutions turned during the flashing interval T. If the wheel were actually turning at the higher frequency $f + mf_s$, then during time T it would turn by $(f + mfs)T = fT + mf_sT = fT + m$ revolutions, that is, it would cover m whole additional revolutions. An observer would miss these extra m revolutions completely. The perceived rotational speed for an observer is always given by $f_a = f \mod (f_s)$. The next example illustrates these remarks.

Example 1.4.11

http://upload.wikimedia.org/wikipedia/commons/5/5e/Strobe_2.gif

6 DSP Frequency Units (1.4.3)

Figure 1.4.4 compares the various frequency scales that are commonly used in DSP, and the corresponding Nyquist intervals. A sampled sinusoid takes the form in these units:

$$e^{j2\pi fTn} = e^{j2\pi(f/f_s)n} = e^{j\Omega Tn} = e^{j\omega n} \tag{14}$$

being expressed more simply in terms of ω. Sometimes f is normalized with respect to the Nyquist frequency $f_N = f_s/2$, that is, in units of f/f_N. In this case, the Nyquist interval becomes $[-1, 1]$. In multirate applications, where successive digital processing stages operate at different sampling rates, the most convenient set of units is simply in terms of f. In fixed-rate applications, the units of ω or f/f_s are the most convenient.

Figure 6: Orfanidis Fig. 1.4.4 Commonly used frequency units..

