1 Lecture Outline

Reading: Chapter 6 Transfer Function

- Linear phase and group delay (6.3)
- Pole/Zero Designs: Parametric Resonators
- Pole/Zero Designs: Notch Filter

2 Linear Phase

Example: Consider the length M, Moving-Average (MA) filter with impulse response

$$h(n) = \frac{1}{M}\delta(n) + \frac{1}{M}\delta(n-1) + \cdots + \frac{1}{M}\delta(n-M+1) \tag{1}$$

We can show that the phase response of the filter is a linear function in ω as follows:

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{M-1} \frac{1}{M} e^{-j\omega n} \\ &= \frac{1}{M}\left[\frac{1-e^{-j\omega M}}{1-e^{-j\omega}}\right] \\ &= \frac{1}{M}\left[\frac{e^{-j\omega M/2}\left(e^{j\omega M/2}-e^{-j\omega M/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}\right] \\ &= \frac{1}{M}\left[\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right] e^{-j\omega(M-1)/2} \\ &= |H(\omega)| e^{j\angle H(\omega)}. \end{aligned} \tag{2}$$

Thus the phase response is *linear* in ω

$$\begin{aligned} \angle H(\omega) &= -\omega(M-1)/2 \\ &= -\omega\alpha. \end{aligned} \tag{3}$$

The *phase delay* is independent of frequency

$$d(\omega) = -\angle H(\omega)/\omega = \alpha = (M-1)/2. \tag{4}$$

Such linear phase filters have the desirable property that every frequency component is delayed, in samples, by the same amount, i.e.

$$e^{j\omega n} \quad \rightarrow \quad |H(\omega)| e^{j\omega(n-\alpha)}. \tag{5}$$

We will show later that FIR filters can easily be designed to have linear phase (symmetry in filter coefficients), however, IIR filters cannot be designed to have linear phase.

3 Group Delay

Definition: The group delay of a filter H is defined as

$$d_g(\omega) = -\frac{d}{d\omega}\angle H(\omega). \tag{6}$$

Figure 1: Ludeman's linear phase response handout and discussion

Group delay measures the *linearity* of phase response. The closer the group delay is to a constant, the closer the phase response is to linear.

Example: The group delay of a linear phase filter is constant (independent of ω). In the case of the ideal delay system

$$\begin{aligned} d_g(\omega) &= -\frac{d}{d\omega}\left(-\alpha\omega\right) \\ &= \alpha \end{aligned} \tag{7}$$

3.1 Transient Response

Students should read the subsection 6.3.2. Transient Response.

4 Pole/Zero Designs: Parametric Resonators

A "resonator" filter has a magnitude response that is dominated by a single, narrow peak at frequency ω_0. This filter "enhances" the band of frequencies around ω_0. To make a peak in the magnitude response at $\omega = \omega_0$, we use a pole with magnitude $0 < R < 1$ at an angle of ω_0. If the frequency response is to have Hermitian symmetry (necessary for processing real-valued signals), the pole must also be accompanied by its complex-conjugate. Therefore our poles are of the form $p_1 = Re^{j\omega_0}$ and $p_1^* = Re^{-j\omega_0}$.

Figure 2: Orfanidis p. 244 Figure 6.4.2 Frequency response of resonator filter.

These poles lead to a transfer function of

$$\begin{aligned} H(z) &= \frac{b_0}{\left(1-p_1 z^{-1}\right)\left(1-p_1^* z^{-1}\right)} \\ &= \frac{b_0}{\left(1-Re^{j\omega_0} z^{-1}\right)\left(1-Re^{-j\omega_0} z^{-1}\right)} \\ &= \frac{b_0}{1-R\left(e^{j\omega_0}+e^{-j\omega_0}\right) z^{-1}+R^2 z^{-2}} \\ &= \frac{b_0}{1+a_1 z^{-1}+a_2 z^{-2}} \end{aligned} \tag{8}$$

where

$$\begin{aligned} a_1 &= -2R\cos\left(\omega_0\right) \\ a_2 &= R^2 \end{aligned} \tag{9}$$

For convenience, we choose b_0 so that $|H(\omega_0)| = 1$ which leads to

$$\left|\frac{b_0}{1-2R\cos(\omega_0)e^{-j\omega_0}+R^2 e^{-j2\omega_0}}\right| = 1 \tag{10}$$

or

$$b_0 = (1-R)\left[1-2R\cos(\omega_0)+R^2\right]^{1/2} \tag{11}$$

The 3dB width $\Delta\omega$ of the peak is defined as the bandwidth at half maximum of the magnitude squared response. $\Delta\omega$ can be solved by determining the 3dB frequencies ω_1, ω_2 such that:

$$\frac{\left|H(\omega_1)\right|^2}{\left|H(\omega_0)\right|^2} = \frac{\left|H(\omega_2)\right|^2}{\left|H(\omega_0)\right|^2} = 1/2 \tag{12}$$

Then for a given $\Delta\omega = \omega_2 - \omega_1$, the 2nd order resonator would have

$$\Delta\omega \approx 2(1-R). \tag{13}$$

The closer R is one, the smaller $\Delta\omega$ becomes and the sharper the peak becomes. However, as R approaches one (pole magnitude approaches one) the slower the transient decays and the longer it takes for the filter to reach steady state.

5 Pole/Zero Designs: Notch Filters

To make a "notch" in the magnitude response at $\omega = \omega_0$ and therefore notch out this frequency from the spectrum, we place a zero pair (for processing real-valued signals) inside the unit circle ($0 < r < 1$) at an angle of $\pm\omega_0$. This leads to a transfer function of

$$\begin{aligned} H(z) &= \left(1-z_1 z^{-1}\right)\left(1-z_1^* z^{-1}\right) \\ &= \left(1-re^{j\omega_0} z^{-1}\right)\left(1-re^{-j\omega_0} z^{-1}\right) \\ &= 1-r\left(e^{j\omega_0}+e^{-j\omega_0}\right) z^{-1}+r^2 z^{-2} \\ &= b_0+b_1 z^{-1}+b_2 z^{-2} \end{aligned} \tag{14}$$

where

$$\begin{aligned} b_0 &= 1 \\ b_1 &= -2r\cos\left(\omega_0\right) \\ b_2 &= r^2 \end{aligned} \tag{15}$$

Equations regarding 3 dB width of the notch are similar to the resonator.

A slight generalization of the resonator places a zero near each pole along the same direction as the pole

$$\begin{aligned} z_1 &= re^{j\omega_0} \\ z_1^* &= re^{-j\omega_0} \end{aligned} \tag{16}$$

where $0 \leq r \leq 1$ in order to "tighten" up the peak or notch in the magnitude response.

Figure 3: Orfanidis p. 249 Figure 6.4.5 Frequency response of resonator filter.

The transfer function is then

$$\begin{aligned} H(z) &= \frac{\left(1 - z_1 z^{-1}\right)\left(1 - z_1^* z^{-1}\right)}{\left(1 - p_1 z^{-1}\right)\left(1 - p_1^* z^{-1}\right)} \\ &= \frac{\left(1 - re^{j\omega_0} z^{-1}\right)\left(1 - re^{-j\omega_0} z^{-1}\right)}{\left(1 - Re^{j\omega_0} z^{-1}\right)\left(1 - Re^{-j\omega_0} z^{-1}\right)} \\ &= \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned} \tag{17}$$

where

$$\begin{aligned} b_1 &= -2r\cos\left(\omega_0\right) \\ b_2 &= r^2 \\ a_1 &= -2R\cos\left(\omega_0\right) \\ a_2 &= R^2 \end{aligned} \tag{18}$$

$r < R$, the pole is closer to the unit circle and dominates the response giving rise to a peak at ω_0.

$r > R$ the zero is closer to the unit circle and dominates the response giving rise to a dip (or notch) at ω_0.

$r = 1$ then there is an exact notch at ω_0.

When $r \approx R$, $|H(\omega)| = 1$ for frequencies distant to ω_0 since the distance from the moving point $e^{j\omega}$ to the pole/zero pairs are nearly the same. Near the vicinity of ω_0, $|H(\omega)|$ will vary dramatically developing a peak or dip.

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