1 Lecture Outline

Reading: Chapter 6 Transfer Function

- Transfer functions (6.2) FIR and IIR cases
- Sinusoidal response (6.3)

Note that we will give a brief review of Sections 6.1 and 6.2 since this chapter is covered in a standard undergraduate course in signals and systems.

2 FIR Transfer Functions

Example 6.2.2: Determine the transfer function, $H(z)$ and impulse response, $h(n)$ for the filter

$$y(n) = x(n) - x(n-4). \tag{1}$$

Assume the filter is causal.

The lack of a feedback term implies the filter is FIR. We begin by taking the z-transform

$$Y(z) = X(z) - z^{-4}X(z) \tag{2}$$

which leads us to

$$Y(z) = \left(1 - z^{-4}\right) X(z) \tag{3}$$

and hence

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= 1 - z^{-4} \end{aligned} \tag{4}$$

The zeros of $H(z)$ are the four roots of unity, i.e. $z^4 = 1$ or $z = 1, j, -1, -j$. These zeros all have a magnitude of one and angles $\omega = 0,\ \pi/2,\ \pi,\ 3\pi/2$. (There are also four trivial poles located at $z = 0$ but these have no influence on the magnitude response.) The pole-zero pattern is shown below. Clearly, the magnitude response will dip (notches in the filter) at these corresponding frequencies (also shown below). Since the zeros are on the unit-circle, the dip will actually go to zero.

Figure 1: Orfanidis p. 226 Pole/zero pattern and magnitude response.

Since the filter is causal, the ROC is $|z| > 0$ and inversion of $H(z)$ yields the impulse response

$$h(n) = \delta(n) + \delta(n-4). \tag{5}$$

The frequency response is

$$H(\omega) = 1 - e^{-j4\omega}. \tag{6}$$

The block diagram realization of the direct form and sample processing algorithm are shown below.

Figure 2: Orfanidis p. 226 Block diagram realization and sample processing algorithm.

3 IIR Transfer Functions

Example 6.2.3a: Determine the transfer function, $H(z)$ and impulse response, $h(n)$ for the filter

$$y(n) = 0.25y(n-2) + x(n). \tag{7}$$

Assume the filter is causal.

The presence of a feedback term suggests the filter is IIR. We begin by taking the z-transform

$$Y(z) = 0.25z^{-2}Y(z) + X(z) \tag{8}$$

or

$$\left[1 - 0.25z^{-2}\right] Y(z) = X(z) \tag{9}$$

which leads us to

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1}{1 - 0.25z^{-2}} \\ &= \frac{0.5}{1 - 0.5z^{-1}} + \frac{0.5}{1 + 0.5z^{-1}} \end{aligned} \tag{10}$$

There are two poles located at $z = 0.5$ and $z = -0.5$ and the pole/zero pattern is shown below. (There are also two trivial zeros located at $z = 0$.) These poles have a magnitude of 0.5 and angles $\omega = 0,\ \pi$.

Figure 3: Orfanidis p. 227 Pole/zero pattern and magnitude response.

The frequency response is

$$H(\omega) = \frac{1}{1 - 0.25e^{-j2\omega}}. \tag{11}$$

Clearly, the magnitude response will be rise at these $\omega = 0,\ \pi$. Since one pole is located at DC ($\omega = 0$) and the other at the Nyquist (or foldover) frequency ($\omega = \pi$), the filter will emphasize (or boost) the low and high frequencies and thus act like a bandstop filter. The precise amount of magnitude response emphasis is found as

$$|H(0)| = |H(\pi)| = 4/3. \tag{12}$$

Since the filter is causal, the ROC is $|z| > 0.5$ and inversion of $H(z)$ yields the impulse response

$$h(n) = 0.5\,(0.5)^n\,u(n) + 0.5\,(-0.5)^n\,u(n). \tag{13}$$

The block diagram realization of the direct form and sample processing algorithm are shown below.

Figure 4: Orfanidis p. 227 Block diagram realization and sample processing algorithm.

Example 6.2.3b: Students should study this example and see how a simple change of sign on the feedback term can dramatically change the frequency response of the filter.

4 Sinusoidal Response

The fundamental theory (eigenfunction theory) of linear systems holds that if

$$x(n) = e^{j\omega_0 n}, \ -\infty < n < \infty \tag{14}$$

then

$$\begin{aligned} y(n) &= H(\omega_0)e^{j\omega_0 n}, \ -\infty < n < \infty \\ &= |H(\omega_0)|\,e^{j(\omega_0 n + \angle H(\omega_0))}. \end{aligned} \tag{15}$$

where $H(\omega_0)$ is the frequency response of the filter at the frequency ω_0. The impact of $H(\omega_0)$ on $e^{j\omega_0 n}$ is to scale the amplitude by $|H(\omega_0)|$ and phase shift by $\angle H(\omega_0)$.

Figure 5: Orfanidis Figure 6.3.1, p. 230.

Example: Consider the ideal delay system

$$y(n) = x(n-m). \tag{16}$$

The impulse and frequency responses are

$$h(n) = \delta(n-m) \quad \leftrightarrow \quad H(\omega) = e^{-j\omega m}. \tag{17}$$

The magnitude and phase responses are

$$|H(\omega)| = 1, \quad \angle H(\omega) = -\omega m. \tag{18}$$

Note that $\angle H(\omega)$ has a phase response which is linear in ω.

Definition: The phase delay of a filter H is defined as

$$d(\omega) \quad = \quad -\frac{\angle H(\omega)}{\omega} \tag{19}$$

The phase delay gives the delay in samples imposed by the filter at a particular frequency.

Example: For the ideal delay system given above, the phase delay is (appropriately) m samples.

Since $\angle H(\omega) = -\omega d(\omega)$, the sinusoidal response of a filter H can be expressed using the phase delay as

$$y(n) \quad = \quad |H(\omega_0)| \, e^{j\omega_0(n-d(\omega_0))}, \; -\infty < n < \infty. \tag{20}$$

Clearly, different frequency components get delayed by different amounts depending on the filter's phase delay.