1 Lecture Outline

Reading: Chapter 6 Transfer Function

- Equivalent descriptions of digital filters (6.1)
- Transfer functions (6.2)

Note that we will give a brief review of Sections 6.1 and 6.2 since this chapter is covered in a standard undergraduate course in signals and systems.

2 Equivalent Descriptions of Digital Filters (6.1)

We are now interested in the relationships between the description of a filter using the transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_L z^{-L}}{1 + a_1 z^{-1} + \cdots + a_M z^{-M}} \tag{1}$$

and the following descriptions

- Impulse response, $h(n)$
- I/O convolutional equation, $y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m)$
- I/O difference equation, $y(n) = -\sum_{k=1}^{M} a_k y(n-k) + \sum_{k=0}^{L} b_k x(n-k)$
- Block diagram realization
- Frequency response, $H(\omega) = \frac{b_0 + b_1 e^{-j\omega} + \cdots + b_L e^{-j\omega L}}{1 + a_1 e^{-j\omega} + \cdots + a_M e^{-j\omega M}}$
- Pole/zero pattern

Figure 1: Orfanidis p. 214 Figure 6.1.1: equivalent descriptions of filters

As we will show, we can translate from one filter description to another. Each description will have its use in DSP.

3 Transfer Functions (6.2)

Translation from one description to another is best illustrated through an example.

Example: Consider the following transfer function of a digital filter

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}. \tag{2}$$

Assume the filter is causal.

1. Since the filter is causal, the ROC is $|z| > 0.8$. To obtain the impulse response we take the inverse z-transform which leads to

$$H(z) = \frac{5}{1 - 0.8z^{-1}} + \frac{2z^{-1}}{1 - 0.8z^{-1}} \quad \leftrightarrow \quad h(n) = 5(0.8)^n u(n) + 2(0.8)^{(n-1)} u(n-1). \tag{3}$$

   Alternately, we may use partial fraction expansion to write

$$H(z) = -2.5 + \frac{7.5}{1 - 0.8z^{-1}} \quad \leftrightarrow \quad h(n) = -2.5\delta(n) + 7.5(0.8)^n u(n). \tag{4}$$

2. Given the impulse response, we can obtain the general I/O convolutional equation for the filter

$$y(n) = -5.0x(n) + 7.5\left[(0.8)x(n-1) + (0.8)^2 x(n-2) + \ldots\right] \tag{5}$$

3. The I/O difference equation can be obtained from $H(z)$ as well. The z-domain equivalent of convolution is

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}} X(z) \end{aligned} \tag{6}$$

   or

$$\left(1 - 0.8z^{-1}\right) Y(z) = \left(5 + 2z^{-1}\right) X(z) \tag{7}$$

   or

$$Y(z) = 0.8z^{-1} Y(z) + 5X(z) + 2z^{-1} X(z) \tag{8}$$

   Taking the inverse z-transform of (8), we arrive at

$$y(n) = 0.8y(n-1) + 5x(n) + 2x(n-1). \tag{9}$$

   For a given $x(n)$, we could implement the filter in MATLAB with:

```
b = [5; 2];
a = [1; -0.8];
y = filter(b,a,x)
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   We see that **b** and **a** are created straight from (1).

4. Given the I/O difference equation we can easily form the direct-form realization as in Figure 6.2.1

5. The sample processing algorithm is given as

Figure 2: Orfanidis p. 217 Figure 6.2.1: Direct form realization of $H(z)$

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for each input sample x do:
   y = 0.8 w1 + 5 x + 2 v1
   v1 = x
   w1 = y
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6. The frequency response $H(\omega)$ can be obtained from $H(z)$ by evaluating z on the unit circle, $z = e^{j\omega}$

$$H(\omega) = \frac{5 + 2e^{-j\omega}}{1 - 0.8e^{-j\omega}}. \tag{10}$$

7. $H(z)$ has a pole at $z = 0.8$ and a zero at $z = -0.4$. Thus the pole/zero pattern is given by

Figure 3: Orfanidis p. 218 Figure 6.2.2 Pole/zero pattern

A simple magnitude response can be sketched from the pole/zero pattern. We have one pole at an angle of $\omega = 0$ and can expect a peak at this frequency. We have one zero at an angle of $\omega = \pi$ and can expect a dip at this frequency. The magnitude responses at these frequencies are easily computed from $H(\omega)$

$$|H(0)| = \left|\frac{5 + 2}{1 - 0.8}\right| = 35, \quad |H(\pi)| = \left|\frac{5 - 2}{1 + 0.8}\right| = 1.67. \tag{11}$$

Note that since the DTFT is computed through evaluation of the z-transform on the unit-circle, i.e. $z = e^{j\omega}$, $|H(z)|_{z=1} = |H(\omega)|_{\omega=0}$ and $|H(z)|_{z=-1} = |H(\omega)|_{\omega=\pi}$.

Figure 4: Orfanidis p. 218 Figure 6.2.2: Magnitude response sketch

3.1 Generalizations

In general, a transfer function will have the form

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_L z^{-L}}{1 + a_1 z^{-1} + \cdots + a_M z^{-M}}. \quad (12)$$

- If the filter is FIR, $D(z)$ is a constant, i.e. there is no feedback component.
- Assuming a_k and b_k are real-valued, then $H(z)$ will have L zeros and M poles. If any of the poles/zeros are complex-valued, then they occur in conjugate pairs.
- To obtain the impulse response $h(n)$ we take the inverse z-transform using one of our techniques. Normally the ROC is specified or implied by virtue of a system property such as causality (ROC extends outward from the outermost pole) or stability (ROC contains the unit circle) or both.
- The I/O difference equation is obtained from $H(z) = Y(z)/X(z)$ as

$$\left(1 + a_1 z^{-1} + \cdots + a_M z^{-M}\right) Y(z) = \left(b_0 + b_1 z^{-1} + \cdots + b_L z^{-L}\right) X(z) \quad (13)$$

  which then leads to

$$y(n) = -a_1 y(n-1) + \cdots + -a_M y(n-M) + b_0 x(n) + b_1 x(n-1) + \cdots + b_L x(n-L). \quad (14)$$