1 Lecture Outline

Reading: Chapter 5 z-Transforms

- Frequency Spectrum (5.4)
- Inverse z-transforms (5.5)

Note that we will give a brief review of Chapter 5 since this chapter is covered in a standard undergraduate course in signals and systems.

2 Frequency Spectrum (5.4)

The frequency spectrum or spectrum or DTFT of a signal $x(n)$ is defined as[1]

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}, \quad \text{Analysis equation} \tag{1}$$

Clearly this is equivalent to the z-transform evaluated on the unit circle, i.e. replacing z with $e^{j\omega}$:

$$\begin{aligned} X(\omega) &= X(z)\,|_{z=e^{j\omega}} \\ &= \sum_{n=-\infty}^{\infty} x(n)z^{-n}\,|_{z=e^{j\omega}} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}. \end{aligned} \tag{2}$$

The frequency spectrum is often denoted $X(e^{j\omega})$ to explicitly indicate evaluation at $z = e^{j\omega}$.

Figure 1: Orfanidis Fig. 5.4.1 p. 199

The *frequency response* $H(\omega)$ of a linear system $h(n)$ with transfer function $H(z)$ is defined by

$$\begin{aligned} H(\omega) &= H(z)\,|_{z=e^{j\omega}} \\ &= \sum_{n=-\infty}^{\infty} h(n)z^{-n}\,|_{z=e^{j\omega}} \\ &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}. \end{aligned} \tag{3}$$

Obviously in order for the spectrum or frequency response to exist, $z = e^{j\omega}$ must be in the ROC.

[1] Some texts use the notation $X(e^{j\omega})$

As a reminder, the digital frequency ω (units of radians/sample) is related to physical frequency, f (Hz) with

$$\omega = 2\pi f/f_s. \tag{4}$$

Thus the Nyquist interval $-f_s/2 \le f \le f_s/2$ is equivalent to $-\pi \le \omega \le \pi$.

2.1 Pole/zero patterns

In general, the system function will be a rational function of the form

$$H(z) = \frac{B(z)}{A(z)}. \tag{5}$$

For FIR DT systems, $A(z) = 1$. In general, IIR systems have a non-trivial $A(z)$.

Example: Consider the following rational transfer function with a single zero and single pole

$$\begin{aligned} H(z) &= \frac{z - z_1}{z - p_1} \\ &= \frac{1 - z_1 z^{-1}}{1 - p_1 z^{-1}} \end{aligned} \tag{6}$$

The corresponding frequency response is obtained by evaluation on the unit circle

$$H(\omega) = \frac{e^{j\omega} - z_1}{e^{j\omega} - p_1} \tag{7}$$

The magnitude response is given by

$$|H(\omega)| = \frac{\left|e^{j\omega} - z_1\right|}{\left|e^{j\omega} - p_1\right|} \tag{8}$$

The quantity $\left|e^{j\omega} - z_1\right|$ is the distance from the point $e^{j\omega}$ to z_1 and $\left|e^{j\omega} - p_1\right|$ is the distance from the point

Figure 2: Orfanidis Fig. 5.4.2 p. 200

$e^{j\omega}$ to p_1. Therefore the magnitude response is the ratio of these distances. As $e^{j\omega}$ moves around the unit circle, the distances and hence ratio will vary.

As $e^{j\omega}$ passes near the pole, $\left|e^{j\omega} - p_1\right|$ gets small and so $|H(\omega)|$ gets large. As $e^{j\omega}$ passes near the zero, $\left|e^{j\omega} - z_1\right|$ gets small and so $|H(\omega)|$ gets small.

2.2 Properties of the DTFT

For real-valued $x(n)$, the quantity $X(\omega)$ is Hermitian (conjugate symmetric)

$$X(\omega)^* = X(-\omega). \tag{9}$$

This implies

$$\begin{aligned} |X(\omega)| &= |X(-\omega)| \\ \arg X(\omega) &= -\arg X(-\omega) \end{aligned} \tag{10}$$

3 Inverse z-transforms (5.5)

The problem of inverting $X(z)$ is to find the signal $x(n)$ whose z-transform is $X(z)$ with the specified ROC. There are four methods which can be used to compute the inverse z-transform:

1. Inspection
2. Power series expansion
3. Partial fraction expansion
4. Complex contour integral

We focus on 1) and 3).

3.1 Inspection

By definition of the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} z^{-n} \tag{11}$$

the coefficient of z^{-k} is the kth term in the sequence $x(n)$. Therefore, if $X(z)$ is in a power series (polynomial) form, the coefficients can be picked off the power series and the sequence can be formed.

Example: Find the inverse z-transform of

$$X(z) = 4 + 5z^{-1} + 6z^{-2} + 7z^{-3} \tag{12}$$

where the ROC is $|z| > 0$. By definition we have

$$x(n) = 4\delta(n) + 5\delta(n-1) + 6\delta(n-2) + 7\delta(n-3) \tag{13}$$

3.2 Partial Fractions

The partial fraction expansion (PFE) method can be applied to z-transforms that are ratios of two polynomials in z^{-1}

$$X(z) = \frac{N(z)}{D(z)}. \tag{14}$$

Assuming $D(z)$ has degree of M, there will be M poles and $D(z)$ can be factored as

$$D(z) = \left(1 - p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right)\ldots\left(1 - p_M z^{-1}\right). \tag{15}$$

The PFE of X(z) is then given by

$$X(z) = Q(z) + \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots + \frac{A_M}{1 - p_M z^{-1}} \tag{16}$$

where $Q(z)$ will either be zero, a constant, or a polynomial in z^{-1} depending on the order of $N(z)$. Each term in the PFE of $X(z)$ is then inverted as either a causal or anticausal sequence according to

$$\begin{aligned} a^n u(n) &\leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a| \\ -a^n u(-n-1) &\leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| < |a| \end{aligned} \tag{17}$$

There are three cases to examine

- Case I: degree$[N(z)] < M$
- Case II: degree$[N(z)] = M$
- Case III: degree$[N(z)] > M$

Case I: degree$[N(z)] < M$
The PFE of $X(z)$ will then be given by (16) (assuming distinct poles) with $Q(z) = 0$.

Example 5.5.2: Let

$$X(z) = \frac{2 - 2.05z^{-1}}{(1 - 0.8z^{-1})(1 - 1.25z^{-1})} \tag{18}$$

Compute $x(n)$ associated with the ROC $0.8 < |z| < 1.25$.

Expanding in partial-fraction form, we have

$$X(z) = \frac{1}{(1 - 0.8z^{-1})} + \frac{1}{(1 - 1.25z^{-1})} \tag{19}$$

The first term has a pole @ $z = 0.8$. Since the ROC extends outward from this pole we invert as a causal sequence:

$$(0.8)^n u(n) \leftrightarrow \frac{1}{1 - 0.8z^{-1}}, \quad |z| > 0.8 \tag{20}$$

The second term has a pole @ $z = 1.25$. Since the ROC extends inward from this pole we invert as an anticausal sequence:

$$-(1.25)^n u(-n-1) \leftrightarrow \frac{1}{1 - 1.25z^{-1}}, \quad |z| < 1.25 \tag{21}$$

Therefore the inverse z-transform in this example is given by

$$x(n) = (0.8)^n u(n) + -(1.25)^n u(-n-1). \tag{22}$$

Case II: degree$[N(z)] = M$
The PFE of $X(z)$ will then be given by (16) (assuming distinct poles) with $Q(z)$ equal to a constant.

Example 5.5.5: Let

$$X(z) = \frac{10 + z^{-1} - z^{-2}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} \tag{23}$$

Compute $x(n)$ associated with the ROC $|z| > 0.5$.

Expanding in partial-fraction form, we have

$$X(z) = 4 + \frac{4}{(1 - 0.5z^{-1})} + \frac{2}{(1 + 0.5z^{-1})} \tag{24}$$

Note $A_0 = X(z)|z = 0$.

Therefore the inverse z-transform in this example is given by

$$x(n) = 4\delta(n) + 4(0.5)^n u(n) + 2(-0.5)^n u(n). \tag{25}$$

Case III: degree$[N(z)] > M$
In Case III, we first divide the polynomial $D(z)$ into $N(z)$, finding the quotient and remainder polynomials degree$[R(z)] < M$

$$N(z) = Q(z)D(z) + R(z). \tag{26}$$

Then we have

$$\begin{aligned} X(z) &= \frac{N(z)}{D(z)} \\ &= \frac{Q(z)D(z) + R(z)}{D(z)} \\ &= Q(z) + \frac{R(z)}{D(z)}. \end{aligned} \tag{27}$$

The term $R(z)/D(z)$ is then expanded as in Case I.