1 Lecture Outline

Reading: Chapter 5 z-Transforms

- Basic Properties (5.1)
- Region of Convergence (5.2)

Note that we will give a brief review of Chapter 5 since this chapter is covered in a standard undergraduate course in signals and systems.

2 Basic Properties (5.1)

Our use of the z-transform will be as a tool for the analysis, design, and implementation of DT systems (digital filters). We will see that working with DT systems amounts to working with polynomials: adding polynomials, multiplying polynomials together, dividing polynomials into each other, factoring polynomials and determining the roots (zeros), etc.... The z-transform will allow us to solve difference equations in much the same way that the Laplace transform allowed us to solve differential equations.

Given a DT signal, $x(n)$, its z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}. \tag{1}$$

The transform is a polynomial in z^{-1} and the coefficients of the polynomial are the corresponding values of the sequence.

Example: The z-transform of $x(n) = \delta(n-N)$ is given by

$$X(z) = z^{-N}, \quad z > 0. \tag{2}$$

The z-transform is a generalization of the DTFT since if we evaluate the z-transform on the unit circle, i.e. $z = e^{j\omega}$ we have

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}. \tag{3}$$

The z-transform of an impulse response gives the *transfer function*

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}. \tag{4}$$

Example: Let

$$h(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) \tag{5}$$

be the impulse response of a DT system. The transfer function is then

$$H(z) = 1 + 2z^{-1} + 3z^{-2}, \quad z > 0 \tag{6}$$

There are three important properties of the z-transform.

1. **Linearity** If a and b are scalars and $X(z)$, $Y(z)$ are z-transforms of the sequences x, y respectively, then

$$ax(n) + by(n) \leftrightarrow aX(z) + bY(z) \tag{7}$$

This can be seen from the definition

$$\begin{aligned}\sum_{n=-\infty}^{\infty} [ax(n) + by(n)]\, z^{-n} &= \sum_{n=-\infty}^{\infty} ax(n)z^{-n} + \sum_{n=-\infty}^{\infty} by(n)z^{-n} \\ &= a\sum_{n=-\infty}^{\infty} x(n)z^{-n} + b\sum_{n=-\infty}^{\infty} y(n)z^{-n} \\ &= aX(z) + bY(z)\end{aligned} \tag{8}$$

2. **Delay** If $x(n) \leftrightarrow X(z)$, then $x(n-D) \leftrightarrow z^{-D}X(z)$. This is can be seen from the definition

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x(n-D)z^{-n} &= \sum_{n=-\infty}^{\infty} x(n-D)z^{-n+D-D} \\ &= z^{-D}\sum_{n=-\infty}^{\infty} x(n-D)z^{-n+D} \\ &= z^{-D}\sum_{k=-\infty}^{\infty} x(k)z^{-k}; \quad k = n-D \\ &= z^{-D}X(z)\end{aligned} \tag{9}$$

3. **Convolution** $y(n) = h(n) \star x(n) \leftrightarrow Y(z) = H(z)X(z)$. We can see this again with the definition

$$\begin{aligned}Y(z) &= \sum_{n=-\infty}^{\infty} y(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h(k)x(n-k)z^{-n} \\ &= \sum_{k=-\infty}^{\infty} h(k)\sum_{n=-\infty}^{\infty} x(n-k)z^{-n+k-k} \\ &= \sum_{k=-\infty}^{\infty} h(k)\sum_{n=-\infty}^{\infty} x(n-k)z^{-n+k}z^{-k} \\ &= \sum_{k=-\infty}^{\infty} h(k)z^{-k}\sum_{m=-\infty}^{\infty} x(m)z^{-m}; \quad m = n-k \\ &= H(z)X(z)\end{aligned} \tag{10}$$

Example: Compute 4637×518 by convolving digit sequences.

3 Region of Convergence (5.2)

It is clear from the definition of the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{11}$$

that if $x(n)$ has infinite duration, certain values of z (complex-valued) might cause the series to diverge.

Definition: The region of convergence (ROC) of $X(z)$ is the subset of the complex z-plane for which $X(z)$ converges, i.e.

$$\text{ROC} = \{z \in \mathcal{C} | X(z) \neq \infty\} \tag{12}$$

The ROC, of course, depends on the signal $x(n)$ being transformed.

Example: Compute the z-transform of the right-sided sequence

$$x(n) \quad = \quad (0.5)^n \, u(n). \tag{13}$$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\ &= \sum_{n=0}^{\infty} (0.5)^n \, z^{-n} \\ &= \sum_{n=0}^{\infty} \left(0.5 z^{-1}\right)^n \end{aligned} \tag{14}$$

If $|0.5z^{-1}| < 1$, then the series converges and we may use the geometric sum formula

$$\sum_{n=0}^{\infty} \alpha^n \quad = \quad \frac{1}{1-\alpha}, \quad |\alpha| < 1 \tag{15}$$

to arrive at a closed form expression

$$X(z) \quad = \quad \frac{1}{1-0.5z^{-1}} \tag{16}$$

The ROC is therefore

$$|0.5z^{-1}| \quad < \quad 1 \tag{17}$$

or

$$|z| \quad > \quad 0.5. \tag{18}$$

Therefore, we write

Figure 1: Orfanidis p. 187

$$(0.5)^n \, u(n) \quad \leftrightarrow \quad X(z) = \frac{1}{1-0.5z^{-1}}, \quad |z| > 0.5. \tag{19}$$

Note that this z-transform has a zero at $z = 0$ and a pole at $z = 0.5$.

Example: Let's now examine the z-transform of the left-sided sequence

$$x(n) = -(0.5)^n u(-n-1). \tag{20}$$

We have

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\ &= -\sum_{n=-\infty}^{-1} (0.5)^n z^{-n} \\ &= -\sum_{n=-\infty}^{-1} \left[(0.5)^{-1} z\right]^{-n} \\ &= -\sum_{n=1}^{\infty} \left[(0.5)^{-1} z\right]^{n} \\ &= 1 - \sum_{n=0}^{\infty} \left[(0.5)^{-1} z\right]^{n} \end{aligned} \tag{21}$$

For convergence of the series we require $|(0.5)^{-1} z| < 1$. Then

$$\begin{aligned} X(z) &= 1 - \frac{1}{1-(0.5)^{-1} z} \\ &= -\frac{(0.5)^{-1} z}{1-(0.5)^{-1} z} \\ &= \frac{1}{1-0.5z^{-1}}. \end{aligned} \tag{22}$$

The ROC is therefore

$$|(0.5)^{-1} z| < 1 \tag{23}$$

or

$$|z| < 0.5. \tag{24}$$

Therefore, we write

Figure 2: Orfanidis p. 188

$$-(0.5)^n u(-n-1) \quad \leftrightarrow \quad X(z) = \frac{1}{1-0.5z^{-1}}, \quad |z| < 0.5. \tag{25}$$

Note that this z-transform has a pole (zero of a denominator polynomial) at $z = 0.5$.

We see that while these two exponential sequences have the same z-transform they have completely disjoint ROCs. In general, for any complex-valued a, the z-transform of exponential sequences is given by ince any

Right-sided (causal),	$a^n u(n) \leftrightarrow \dfrac{1}{1 - az^{-1}}, \quad \lvert z\rvert > \lvert a\rvert$	ROC extends outward from outermost pole
Left-sided (anti-causal),	$-a^n u(-n-1) \leftrightarrow \dfrac{1}{1 - az^{-1}}, \quad \lvert z\rvert < \lvert a\rvert$	ROC extends inward from innermost pole

Figure 3: Orfanidis p. 188

ratio of two polynomials can be decomposed into partial fractions, these z-transforms will form the building blocks for more complicated z-transforms.

Example (5.2.3): Find the z-transform and corresponding ROCs for the following sequences

1) (Causal) $x(n) = (0.8)^n u(n) + (1.25)^n u(n)$
2) (Mixed) $x(n) = (0.8)^n u(n) - (1.25)^n u(-n-1)$
3) (Anti-causal) $x(n) = -(0.8)^n u(-n-1) - (1.25)^n u(-n-1)$
4) (Mixed) $x(n) = -(0.8)^n u(n) + (1.25)^n u(n)$

We use the z-transforms of the exponential sequences. See solution p. 192.