1 Lecture Outline

Reading: Chapter 12

- Lloyd's Algorithm
- Introduction to Vector Quantization (VQ)
- VQ Optimization Problem
- k-Means Algorithm
- VQ Principles

2 Lloyd's Algorithm

The scalar quantizer problem centers around optimal (MMSE) solutions for *decision boundaries* and *reconstruction levels*. Obtaining the solution may be difficult and requires knowledge of the pdf of the input process. For input which has a uniform pdf, the optimal quantizer is the uniform quantizer; for input with a non-uniform pdf, the optimal quantizer is nonuniform. Companders transform non-uniformly distributed signals to approximately uniformly-distributed signals; use of a uniform quantizer on companded signal can improve SNR.

Lloyd proposed an iterative approach (suboptimal) to finding decision boundaries and reconstruction levels.

Step 1: Collect a set of speech training data and pick a distance measure (usually mean-squared error)

Step 2: Pick a set of M reconstruction levels, $\{\hat{x}_1, \cdots, \hat{x}_M\}$

Step 3: Group (cluster) samples by minimum distance to reconstruction levels

Step 4: Average speech samples in each cluster to get the updated reconstruction levels, $\{\hat{x}_1, \cdots, \hat{x}_M\}$

Step 5: Compute new decision boundaries, $\{x_0, \cdots, x_M\}$ as the average of adjacent reconstruction levels

Step 6: If reconstruction levels change, goto step 3. Otherwise we have convergence.

Lloyd's algorithm for 1D samples can be extended to multi-dimensional data vectors and is known as the k-means algorithm.

Demo: Lloyd's algorithm for 1D data.

3 Introduction to Vector Quantization

We now consider a generalization of scalar quantization in which a *block* of scalars is coded as a vector, rather than individually. This generalization is referred to as vector quantization (VQ). In VQ we exploit correlations (similarity) in the speech signal to reduce the bit rate.

Consider an $N \times 1$ vector

$$\mathbf{x} = \left[x^{(1)}, x^{(2)}, \cdots, x^{(N)}\right]^T. \tag{1}$$

With quantization, the vector $\mathbf{x}$ is mapped to one of M, N-dimensional vectors or reconstruction levels, $\mathbf{r}_i$

$$\begin{aligned}\hat{\mathbf{x}} &= \mathrm{VQ}\left[\mathbf{x}\right] \\ &= \mathbf{r}_i, \quad 1 \leq i \leq M \\ &= \left[\hat{x}^{(1)}, \hat{x}^{(2)}, \cdots, \hat{x}^{(N)}\right]^T \end{aligned} \tag{2}$$

where we assume $\mathbf{x} \in C_i$ i.e.,the ith cell or boundary. The reconstruction levels, $\mathbf{r}_i$ are also called *codewords* (as in the scalar case) and the set of codewords, $\{\mathbf{r}_i\}$ the *codebook*. As one might think, the reconstruction level is located at the centroid of the cell, C_i which can have arbitrary size and shape.

Note: $x^{(1)} - \hat{x}^{(1)}$ could be very different than $x^{(2)} - \hat{x}^{(2)}$. The goal of VQ is to reduce the *collective* quantization error.

Example: Consider vector quantization of pairs of samples (2×1 vectors). Assume $M = 6$ reconstruction levels.

Figure 1: Figure 12.15: Vector quantization

4 VQ Optimization Problem

Vector quantization noise is given by

$$\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x} \tag{3}$$

and the cost function or distortion measure (which we want to minimize) is given by

$$J = E\left[\mathbf{e}^T \mathbf{e}\right] \tag{4}$$

where E is the expectation operator. Using the noise equation and definition of expectation we have

$$\begin{aligned} J &= E\left[\left(\hat{\mathbf{x}} - \mathbf{x}\right)^T \left(\hat{\mathbf{x}} - \mathbf{x}\right)\right] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\hat{\mathbf{x}} - \mathbf{x}\right)^T \left(\hat{\mathbf{x}} - \mathbf{x}\right) p_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \\ &= \sum_{i=1}^{M} \int \cdots \int_{\mathbf{x} \in C_i} \left(\mathbf{r}_i - \mathbf{x}\right)^T \left(\mathbf{r}_i - \mathbf{x}\right) p_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \end{aligned} \tag{5}$$

where $p_{\mathbf{x}}(\mathbf{x})$ is the N-dimensional pdf. We seek to minimize the distortion, J by optimally selecting cell boundaries, C_i and reconstruction levels, $\mathbf{r}_i$.

There are two necessary constraints for optimization:

Constraint 1: The vector $\mathbf{x}$ must be quantized to a reconstruction level $\mathbf{r}_i$ that gives the *smallest distortion* between $\mathbf{x}$ and $\mathbf{r}_i$. This implies that we can VQ without explicitly knowing the cell boundaries (although we may have a large search space).

Constraint 2: Each reconstruction level, $\mathbf{r}_i$ must be the *centroid* of the corresponding cell (decision region), C_i.

This is usually a difficult optimization problem to solve and furthermore, we'd like to avoid having to know the joint pdf.

5 k-Means Algorithm

The k-means algorithm developed by Forgy is the multidimension version of Lloyd's algorithm for VQ. This algorithm seeks to iteratively determine cell boundaries, C_i and k reconstruction levels, $\mathbf{r}_i$ which (nearly) minimizes the distortion.

Step 1: Collect a set of training vectors, $\{\mathbf{x}_i\}$ and pick a distance measure (usually mean-squared error).

Step 2: Pick a set of M reconstruction levels, $\{\mathbf{r}_i\}$.

Step 3: Group (cluster) vectors by minimum distance to reconstruction levels.

Step 4: Average vectors in each cluster to get centroid for the cluster. Update reconstruction levels, $\{\mathbf{r}_i\}$ with the value of the centroid.

Step 5: If reconstruction levels change, goto step 3. Otherwise we have convergence. The result minimizes the total distortion over the entire set of training data.

Figure 2 illustrates cluster formation and selection of centroid as the reconstruction level.

Figure 2: Figure 12.16: k-means algorithm

Demo: k-means algorithm for 2D data.

6 VQ Principles

- VQ exploits correlations among the samples in the vector. Any transformation which reduces these correlations, reduces the VQ advantage.
- An advantage of VQ over scalar quantization is that it can reduce the number of reconstruction levels when the distortion is held constant and thus reduce the bit rate. Alternately, when the number of reconstruction levels if fixed, it can reduce the distortion.

Figure 3 illustrates a generic transmitter/receiver based on VQ.

Figure 3: Block diagram of a vector quantizer