# 1 Lecture Outline

**Reading: Chapter 12**

- Introduction
- Non-uniform quantization
- Companding and $\mu$-law quantizers
- Adaptive quantization
- Differential quantization

# 2 Review of Random Processes

We will first provide a brief review of probability and random process theory. We denote a probability density function (pdf) as $p_X(x)$ where $X$ denotes the random variable and $x$ denotes the value $X$ takes (realization).

For the pdf $p_X(x)$, the probability that $a \leq X \leq b$ is given by

$$P\{a \leq X \leq b\} = \int_a^b p_X(x)dx. \tag{1}$$

From this definition we have

$$\int_{-\infty}^{+\infty} p_X(x)dx = 1 \tag{2}$$

and

$$P\{X \leq b\} = \int_{-\infty}^{b} p_X(x)dx. \tag{3}$$

In this class, we will work with the following pdfs.

- Uniform

$$p_X(x) = \begin{cases} \beta - \alpha, & \alpha \leq x \leq \beta \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

- Gaussian

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \tag{5}$$

  where $\mu$ is the mean of the distribution (see below) and $\sigma^2$ is the variance of the distribution.

- Laplacian

$$p_X(x) = \frac{1}{2b} \exp\left\{-\frac{|x-\mu|}{b}\right\} \tag{6}$$

  where $\mu$ is the mean of the distribution (see below) and $b/\sqrt{2}$ is the variance of the distribution.

The mean of a distribution is defined as

$$E[X] \quad = \quad \int_{-\infty}^{\infty} x p_X(x) dx. \tag{7}$$

For a uniform pdf,

$$E[X] \quad = \quad 1/(\beta - \alpha). \tag{8}$$

The variance $\sigma^2$ of a distribution is defined as

$$\sigma^2 \quad = \quad E[(X - \mu)^2]. \tag{9}$$

For a uniform pdf

$$\sigma^2 \quad = \quad (\beta - \alpha)^2/12 \tag{10}$$

and Laplacian pdf

$$\sigma^2 \quad = \quad 2b^2. \tag{11}$$

## 3 Non-uniform Quantization

Basic idea:

Figure 1: Various pdfs with uniform quantization and non-uniform quantization

Max first published his work in optimal scalar quantization in J. Max, "Quantizing for Minimum Distortion," *IRE Trans. Information Theory*, Mar. 1960. The optimal quantization problem is as follows.

"For a random variable $x$ with a known pdf, find the $M$ quantizer levels (both decision boundaries, $x_i$ and reconstruction levels, $\hat{x}_i$) that minimizes

$$J \quad = \quad E\left[(\hat{x} - x)^2\right] \tag{12}$$

where $E$ denotes the expectation operator and $\hat{x}$ is the quantized version of $x$." (This optimization problem seeks to minimize the quantization noise energy for a fixed number of quantizer bits. Equivalently, for a fixed SNR, we would like to minimize the number of bits in the codeword.)

Figure 2: Figure 12.8: Nonuniform quantization

Suppose $x$ is a random variable denoting a sample of the speech signal, $x[n]$ with pdf, $p_x(x)$. Then (12) becomes

$$\begin{aligned} J &= \int_{-\infty}^{\infty} p_x(x) \left(\hat{x} - x\right)^2 dx \\ &= \sum_{i=1}^{M} \int_{x_{i-1}}^{x_i} p_x(x) \left(\hat{x}_i - x\right)^2 dx \end{aligned} \tag{13}$$

where in the last line we have noted that since $\hat{x}$ is one of the $M$ reconstruction levels, we can divide the integral up into contributions to $J$ over each decision interval (see Fig. 2).

In order to minimize the MSE, we have to choose optimal decision boundaries $x_i$ and reconstruction levels $\hat{x}_i$ to satisfy both

$$\frac{\partial J}{\partial x_k} = 0, \quad 1 \leq k \leq M-1 \tag{14}$$

and

$$\frac{\partial J}{\partial \hat{x}_k} = 0, \quad 1 \leq k \leq M-1 \tag{15}$$

where we assume $x_0 = -\infty$ and $x_M = \infty$ and do not include them in our criteria (see Fig. 2).

It can be shown (see text) that the decision boundaries in (14) are satisfied with

$$x_k = \frac{\hat{x}_{k+1} + \hat{x}_k}{2}, \quad 1 \leq k \leq M-1 \tag{16}$$

or in other words, the optimal decision level $x_k$ is chosen as the average of the reconstruction levels $\hat{x}_k$ and $\hat{x}_{k+1}$.

Likewise the reconstruction levels in (15) are satisfied with

$$\hat{x}_k = \frac{\int_{x_{k-1}}^{x_k} p_x(x) x dx}{\int_{x_{k-1}}^{x_k} p_x(x) dx}, \quad 1 \leq k \leq M \tag{17}$$

or in other words, the optimal reconstruction level $\hat{x}_k$ is chosen as the *centroid* (see Fig. 6) of $p_x(x)$ over the interval $x_{k-1} \leq x \leq x_k$. The numerator is the mean over the interval $(x_{k-1}, x_k)$ and the denominator (normalizing factor) is the probability of $x$ over the same interval.

Figure 3: decision boundaries

Figure 4: Figure 12.9: Centroid of pdf

Figure 5: uniform quantizer for uniform distribution

Of course, the solutions are coupled with one another and require knowledge of the pdf of $x$. Solving for the actual values of $x_k$ and $\hat{x}_k$ is a non-linear problem and can be cumbersome for some pdfs. However, **for the signal with a uniform pdf, the uniform quantizer can be shown to be optimal.**

Paez and Glisson solved for the optimum quantizers for Laplace and gamma distributions both of which (as we have seen) are good fits to actual speech distributions. See M. Paez and T. Glisson, "Minimum Mean-Squared Error Quantization in Speech," *IEEE Trans. Communications*, Apr. 1972.

Figure 6: Figure 12.6 and 12.2.

# 4 Companding and $\mu$-Law Quantizers

An alternate strategy for optimal quantizer design, seeks to achieve the effect of nonuniform quantization not by designing the quantizer to match the signal, but to transform the signal (based on its pdf) to match a uniform quantizer. A follow-on to this approach (proposed by Lloyd) does not require an explicit pdf.

This alternate strategy is illustrated in Fig. 7. In the coding stage, a nonlinearity, $T$ is applied to the waveform $x[n]$ to form a new signal $T\{x[n]\} = g[n]$ whose pdf is uniform. A uniform quantizer is then applied giving $\hat{g}[n]$.

Figure 7: Figure 12.10: Companding

Intuitively, we need a transform which brings large values down (COMpresses) and small values up (exPANDS) or *compands* so that we even out (make more uniform) the occurrences of sample values, i.e. dynamic range compression. Such a transform is given by

$$T(x[n]) = \begin{cases} \int_{-\infty}^{x[n]} p_x(\beta)d\beta - \frac{1}{2}, & -\frac{1}{2} \leq g[n] \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

Equation (18) behaves as follows

$$T(x[n]) \approx \begin{cases} \frac{1}{2}, & x[n] \text{ is large} \\ -\frac{1}{2}, & x[n] \text{ is } -\text{large} \\ 0, & x[n] \text{ is small.} \end{cases} \tag{19}$$

We note that this alternative (Fig. 12-10) is not optimal in the MSE sense. The optimal approach is (16) and (17).

Other nonlinearities (such as a logarithm) approximate the companding operation, but are easier to implement and do not require a prior pdf measurement of the signal. The $\mu$-law transformation approximates the logarithm operation but avoids the infinite dynamic range of the log. The $\mu$-law compander, ubiquitous in waveform encoding, is given by

$$T(x[n]) = x_{\max} \frac{\log(1 + \mu \frac{|x[n]|}{x_{\max}})}{\log(1+\mu)} \text{sign}(x[n]) \tag{20}$$

where $\mu$ is a design variable. A typical value is $\mu = 100$. Equation (20) behaves as follows

$$T(x[n]) \approx \begin{cases} \pm x_{\max}, & x[n] \text{ is } \pm \text{large} \\ 0, & x[n] \text{ is small.} \end{cases} \tag{21}$$

Table 1 shows an SNR comparison of various 3 bit quantizers for different input distributions. The quantizers are designed for the given pdfs. In the case of the uniform quantizer, the quantization steps are equal (uniform) but the size depends on the pdf and how "heavily-tailed" the pdf is. For a signal with a uniform pdf, a 3 bit uniform quantizer would yield approximately 18 dB SNR. In the case of the nonuniform quantizer, the quantization steps are not equal (nonuniform) and the decision boundaries and quantization levels depend on the pdf. Finally, the Laplace and Gamma pdfs approximate the speech pdf.

Table 1: Signal-to-Noise Ratios for 3 bit Quantizers (Noll)

| Probability density function used in design of quantizer | Uniform Quantizer SNR (dB) specified pdf / speech input | Nonuniform Quantizer SNR (dB) specified pdf / speech input |
|---|---|---|
| Uniform | 18.0 / – | 18.0 / – |
| Gaussian | 14.3 / 6.7 | 14.6 / 7.3 |
| Laplace | 11.4 / 7.4 | 12.6 / 9.9 |
| Gamma | 11.5 / – | 11.5 / – |
| Speech | 8.4 / 8.4 | 12.1 / 12.1 |
| $\mu$-law ($x_{\max} = 8$, $\mu = 100$) | – / – | – / 9.5 |

# 5 Adaptive Quantization

We are confronted with a dilemma in quantizing speech signals. On the one hand, we wish to choose the quantization step size large enough to accommodate the maximum peak-to-peak range of the signal. On the other hand we would like to make the quantization step size small so as to minimize the quantization noise. This is compounded by the nonstationary nature of speech. One approach to accommodate these amplitude fluctuations is to use a nonuniform quantizer. An alternate approach is to *adapt* the properties of the quantizer to the level of the input signal. Such a quantization method is known as adaptive quantization or adaptive pulse code modulation (APCM).

The basic idea of adaptive quantization is to let the quantization step size in the case of the uniform quantizer or (more generally) the quantizer levels and ranges in the case of the nonuniform quantizer, vary so as to match the variance of the input signal [see Fig. 8(a)]. An alternate point of view is to consider a fixed quantizer characteristic preceded by a time-varying gain which attempts to keep the signal variance constant [see Fig. 8(b)]. In the case of the uniform quantizer, the step size increases/decreases with increased/decreased input signal variance. In the case of the nonuniform quantizer, quantization levels and ranges are scaled linearly to match the signal variance. This results from the fact that a change in variance is simply a scaling of the input signal, i.e., if $E\{x[n]^2\} = \sigma_x^2$ then $E\{(\beta x[n])^2\} = \beta^2\sigma_x^2$.

Figure 8: Figure 5.22 p. 198 Rabiner and Schafer

Consider the design of an adaptive, nonuniform quantizer. To begin, we first design a quantizer based on the pdf of a short segment (20–40ms) of speech. Such pdfs turn out to be single peaked as in the Laplace or gamma pdfs but are actually more accurately described by a Gaussian pdf. Now, assume the short-time pdf is Gaussian with an unknown variance $\sigma_x^2$

$$p_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp(-\frac{x^2}{2\sigma_x^2}). \tag{22}$$

If we can measure the local variance $\sigma_x^2$, then we can *adapt* a nonuniform quantizer to the local pdf.

We note that only one, nonuniform quantizer need be designed for a Gaussian pdf with unity variance since (as noted above), a time-varying gain can be applied to the signal according to the estimated variance.

The feed-forward approach to estimating the variance, assumes the variance is proportional to the short-time signal energy. A primitive estimator is

$$\sigma_x^2[n] = \frac{1}{M} \sum_{m=n}^{n+M-1} x^2[n] \tag{23}$$

while a more general estimator is

$$\sigma_x^2[n] = \beta \sum_{m=-\infty}^{\infty} x^2[n]h[n-m] \tag{24}$$

where $h[n]$ is a LPF and $\beta$ is a constant. The LPF serves to locally "smooth" the variation. The bandwidth of the time-varying variance $\sigma_x^2[n]$ is controlled by the time width of the filter $h[n]$.

Table 2 shows an SNR comparison of various 3 bit adaptive quantizers for speech input. The quantizers are designed for either a Gaussian or Lapacian pdf with an estimated variance. Note for the Laplace pdf (similar to the speech pdf) the 3 bit non-uniform, adaptive quantizer has a SNR of 13.3 dB compared to a SNR of 7.4 dB for the uniform, non-adaptive quantizer.

Table 2: Signal-to-Noise Ratios for 3 bit Adaptive Quantizers with speech input (Noll)

| Probability density function used in design of quantizer | Uniform / Non-uniform nonadaptive quantizer SNR (dB) | Uniform / Non-uniform adaptive ($M = 128$) quantizer SNR (dB) |
|---|---|---|
| $\mu$-law ($x_{\max} = 8$, $\mu = 100$) | 9.5 | – |
| Gaussian | 6.7 / 7.3 | 14.7 / 15.0 |
| Laplace | 7.4 / 9.9 | 13.4 / 13.3 |

# 6 Differential Quantization

Up to now we have investigated instantaneous quantization where *individual* samples are quantized. We have seen, however, that speech is highly correlated both on a short-time scales ($\approx$ 1 ms or 10-15 samples @ 8 kHz) and on a long-time scales (e.g., a pitch period). We can exploit the short-time correlation to improve coding performance.

Short-time correlation implies that neighboring speech samples are "self-similar" and thus not changing too rapidly from one another. The *difference* between adjacent samples should, therefore, have a lower variance than the variance of the signal itself, thus making more effective use of quantization levels. More generally, we can consider predicting the next sample from the previous ones and finding the best prediction coefficients to yield a minimum mean-squared prediction error (as in Chapter 5). We can use in the coding scheme a *fixed* prediction filter to reflect the average correlation of a signal, or we can allow the predictor to short-time *adapt* to the signal's local correlation. In the latter case, we need to transmit the quantized prediction coefficients as well as the prediction error. One particular prediction error coding scheme is illustrated in Fig. 6.

Figure 9: Figure 12.12 Differential coder/decoder

The prediction error signal, $r[n]$ is also referred to as the residual and this quantization approach is referred to as *residual coding*. The quantizer can be of any type, e.g., fixed, adaptive, uniform, or nonuniform. In any case, the parameters are adjusted to match the variance of $r[n]$. Observe that this differential quantization approach can be applied not only to the speech signal itself, but also to parameters that represent the speech, e.g. linear prediction coefficients.

We write the quantized residual as

$$\hat{r}[n] = r[n] + e[n] \tag{25}$$

and the quantized input as

$$\begin{aligned}\hat{x}[n] &= \tilde{x}[n] + \hat{r}[n] \\ &= \tilde{x}[n] + r[n] + e[n] \\ &= \tilde{x}[n] + x[n] - \tilde{x}[n] + e[n] \\ &= x[n] + e[n].\end{aligned} \tag{26}$$

Therefore, the quantized signal samples differ from the input only by the quantization error $e[n]$, which is the quantization error of the residual. If the prediction of the signal is accurate, the variance of $r[n]$ will be smaller than the variance of $x[n]$ so that a quantizer with a given number of levels can be adjusted to give a smaller quantization error than would be possible when quantizing the signal directly.

The differential coder when using a fixed predictor and fixed quantization, is referred to as differential PCM (DPCM). Although this scheme can improve SNR, the improvement is not dramatic. Rather DPCM with both adaptive prediction and adaptive quantization, referred to as ADPCM yields the greatest gains in SNR for a fixed bit rate. CCITT G.721 with toll quality speech ($\approx$ 70 dB SNR) at 32kbps (8000 samples $\times$ 4 bits/sample) has been designed based on ADPCM.

 