Introduction to Wavelets

Taken from *Introduction to Wavelets and Wavelet Transforms* by C. Burrus, R. Gopinath, and H Guo.

A wave is usually defined as an oscillating function of time such as a sinusoid. Fourier analysis is *wave* analysis in that we expand signals in terms of sinusoids (complex exponentials)

$$X[k] = \frac{1}{T}\int_T x(t)\, e^{-jk\Omega_0 t}\, dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\Omega_0 t}.$$

In Fourier analysis, our wave oscillates with equal amplitude over $-\infty \le t \le \infty$ and therefore has infinite energy.

A wavelet is a "small wave" which has finite energy concentrated around a point in time to give a tool for he analysis of transient, nonstationary, or time-varying phenomena. It still has the oscillating wave-like characteristic but also has the ability to allow simultaneous time and frequency analysis.

Figure 1.1

We use wavelets in a series expansion of signals much the same way Fourier analysis used the wave or sinusoid to represent a signal.

Wavelets and Wavelet Expansion Systems

We wish to analyze a signal by expressing it as a linear decomposition

$$f(t) = \sum_{l} a_l \psi_l(t) \tag{1.1}$$

where l is an integer index for the finite or infinite sum, a_l are the real-valued expansion coefficients, and the $\psi_l(t)$ are a set of real-valued functions of t called the expansion set. If the expansion is unique the set is called a basis for the class of functions that can be so expressed.

If the basis is orthogonal

$$\langle \psi_k(t), \psi_l(t) \rangle = \int \psi_k(t)\psi_l(t)\, dt = 0,\ k \ne l \tag{1.2}$$

then the coefficients can be calculated by the inner product

$$a_k = \langle f(t), \psi_k(t) \rangle = \int f(t)\psi_k(t)\, dt. \tag{1.3}$$

If the basis is not orthogonal, then a dual basis exists such that using (1.3) with the dual basis gives the desired coefficients.

For a Fourier series, the orthogonal basis function the $\psi_k(t)$ are $\sin(k\Omega_0 t)$ and $\cos(k\Omega_0 t)$ with frequencies $k\Omega_0$.

For a Taylor series, the nonorthogonal basis functions are the simple monomials t^k.

For a wavelet expansion, a two-parameter system is constructed such that (1.1) becomes

$$f(t) = \sum_{\kappa} \sum_{\varphi} \alpha_{\varphi,\kappa} \psi_{\varphi\kappa}(\tau) \tag{1.4}$$

where both j and k are integer indices and the $\psi_{j,k}(t)$ are the wavelet expansion functions that usually form an orthogonal basis. The set of expansion coefficients $a_{j,k}$ are called the discrete wavelet transform (DWT) of $f(t)$ and (1.4) is the inverse transform.

There are many different wavelet expansion sets.

Specific Characteristics of Wavelet Systems

1. All so-called first-generation wavelet systems are generated from a single scaling function or wavelet by simple scaling and translation. The two-dimensional parameterization is achieved from the function $\Psi(t)$ (mother wavelet) by

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \tag{1.5}$$

where j and k are integers and the factor $2^{j/2}$ maintain constant norm independent of scale j. This parameterization of the time or space location by k and the frequency or scale by j turns out to be extraordinarily effective.

2. Almost all wavelet systems also satisfy the *multiresolution* conditions. This means that if a set of signals can be represented by a weighted sum of $\Psi(t - k)$, then a larger set (including the original) can be represented by a weighted sum of $\Psi(2t - k)$. In other words, if the basic expansion signals are made half as wide and translated in steps half as wide, they will represent a larger class of signals exactly or give a better approximation of any signal.

Figure 1.2: Translation and Scaling of a Wavelet

As the index k changes the location of the wavelet moves which allows the expansion to explicitly represent the location of events in time. As the index j changes, the shape of the wavelet changes in scale. This allows a representation of detail or resolution. Note that as the scale becomes finer (large j), the steps in time become smaller. It is both the narrower wavelet and the smaller steps that allow representation of greater detail or higher resolution.

Wavelet analysis is well-suited to transient signals. Fourier analysis is appropriate to periodic signals or for functions whose statistical characteristics do not change with time. It is the localizing property of wavelets that allow a wavelet expansion of a transient event to be modeled with a small number of coefficients.

Haar Scaling Functions and Wavelets

The multiresolution formulation needs two closely related basic function. In addition to the wavelet $\Psi(t)$ we need another basic function called the *scaling function*, $\varphi(t)$. The simplest possible orthogonal wavelet system is generated from the Haar scaling function and wavelet.

Figure 1.3

Our wavelet representation of signals is then

$$f(t) = \sum_{\kappa=-\infty}^{\infty} \chi_\kappa \varphi(\tau - \kappa) + \sum_{\kappa=-\infty}^{\infty} \sum_{\varphi=-\infty}^{\infty} \delta_{\varphi\,\kappa} \psi(2^{\varphi}\tau - \kappa)\ . \tag{1.6}$$

Figure 1.4

Why is Wavelet Analysis Effective?

1. The size of the wavelet expansion coefficients $a_{j,k}$ or $d_{j,k}$ drop off rapidly with j and k for a large class of signals. This property is why wavelets are effective in signal and image compression, denoising, and detection.

2. The wavelet expansion allows a more accurate local representation and separation of signal characteristics. A Fourier coefficient represents a component that lasts for all time.

3. Wavelets are adjustable and adaptable. Because there is not just one wavelet, they can be designed to fit individual applications.

Multiresolution Analysis

The scaling function $\varphi(t)$ can be expressed in terms of a weighted sum of shifted $\varphi(2t)$

$$\varphi(t) = \sum_n h(n)\sqrt{2}\phi(2t - n) \tag{2.13}$$

Example (p.13 Haar) The Haar scaling function satisfies

$$\varphi(t) = \phi(2t) + \phi(2t - 1) \tag{2.14}$$

which means $h(0) = 1/\sqrt{2},\ \eta(1) = 1/\sqrt{2}$

Figure 2.2

The wavelet $\Psi(t)$ can be expressed in terms of a weighted sum of shifted $\varphi(2t)$

$$\psi(t) = \sum_n h_1(n)\sqrt{2}\phi(2t - n) \tag{2.24}$$

Example (p.16 Haar) The Haar wavelet satisfies

$$\psi(t) = \phi(2t) - \phi(2t - 1)$$

which means $h_1(0) = 1/\sqrt{2},\ \eta_1(1) = -1/\sqrt{2}$

Figure 2.4

We note the scaling function acts as a low-resolution or course approximation and the wavelet provides as a high-resolution or detail

$$f(t) = \underbrace{\sum_{k=-\infty}^{\infty} c_k \varphi(t-k)}_{\text{course}} + \underbrace{\sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} d_{j,k} \psi(2^j t - k)}_{\text{detail}} \tag{2.28}$$

The h_0 (scaling function) and h_1 (wavelet) coefficients provide lowpass and high pass filters for determining the expansion coefficients.

Figure 2.5

Figure 2.6 + 2.7

Figure 2.8

Figure 2.9 + 2.10