Discrete-Time Systems

Definition: A ***DT (digital) system*** is a transformer or operator, $T\{\bullet\}$ that maps an input sequence (input sequence whose values come from a finite set) **x** or $\{x[n]\}$ into an output sequence (output sequence whose values come from a finite set) $\{y[n]\}$

$$y[n] = T\{x[n]\}.$$

The value of the output sequence at each index n may be a function of $x[n]$ at various (even all) values of n.

Figure: I/O Mapping

Example: The *Ideal Delay* system is defined by

$$y[n] = T\{x[n]\} = x[n - n_d]$$

where n_d is a fixed positive (negative) integer called the delay (advance) of the system. The ideal delay (advance) simply shifts the input sequence to the right (left) by n_d samples to form the output.

Example: The *Moving Average* (MA) (N points) system is defined by

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

This system computes the n^{th} sample of the output sequence as the average of N samples of the input sequence around the n^{th} sample. Also called a MA or rectangular window filter.

Linear Systems

Definition: If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is ***linear*** if and only if

$$\begin{aligned} T\{x_1[n] + x_2[n]\} &= T\{x_1[n]\} + T\{x_2[n]\} \\ &= y_1(n) + y_2(n) \end{aligned}$$ (additivity property)

and

$$\begin{aligned} T\{ax[n]\} &= aT\{x[n]\} \\ &= ay[n] \end{aligned}$$ (scaling property).

The above properties can be combined into the principle of superposition

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}.$$

Example: The ideal delay system is linear

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= ax_1[n - n_d] + bx_2[n - n_d] \\ &= aT\{x_1[n]\} + bT\{x_2[n]\} \end{aligned}$$

Time (shift) Invariant

Definition: A DT system is ***time*** **(**or ***shift*****)** ***invariant*** **(*****SI*****)** if a shift of the input sequence produces a corresponding shift in the output sequence

$$T\{x[n-n_d]\} = y[n-n_d]$$

Example: The *D*-fold downsampler is defined by

$$y(n) = T\{x[n]\} = x[nD].$$

The downsampler retains every *D*th sample and discards the *D*–1 in between. This system is not SI. Consider

$$x(n) = 1, 2, 3, 4, 5, \ldots$$

and $D = 2$. Then

$$y(n) = 1, 3, 5, \ldots$$

Now let

$$\begin{aligned} x'(n) &= x(n-1) \\ &= 0,1,2,3,4,5,\ldots \end{aligned}$$

then

$$\begin{aligned} y'(n) &= 0,2,4,\ldots \\ &\neq y(n-1) \end{aligned}.$$

We can prove the general case in the following way.

$$\begin{aligned} T\{x[n-n_d]\} &= x[Dn-n_d] \\ &\neq x[Dn-Dn_d] \\ &= x[D(n-n_d)] \\ &= y[n-n_d] \end{aligned}.$$

Causal

Definition: A DT system is ***causal*** if for every choice of n_0 the output sequence value at $n = n_0$ depends only on the input sequence values for $n \le n_0$.

Example: The backward difference system defined by

$$y[n] = x[n] - x[n-1]$$

is causal since output at index *n* depends only on the inputs at index *n* (present) and $n - 1$ (past).

BIBO Stable

Definition: A sequence is ***bounded*** if there exists a finite *M* such that

$$|x[n]| < M$$

for all *n*.

Definition: A DT system is stable in the bounded input-bounded output (***BIBO***) sense if and only if every bounded input sequence, $x[n]$ produces a bounded output sequence, $y[n]$.

For a system to be BIBO stable we must show that *every* bounded input sequence leads to a bounded output.

LSI Systems

Lemma: Every DT linear system can be completely characterized by its responses to delayed unit-pulse sequences.

Proof: Let $T\{\delta[n-k]\} = h_k[n]$ be the response of the system to $\delta(n - k)$. Then

$$\begin{aligned} y[n] &= T\{x[n]\} \\ &= T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \quad ; x[k] \text{ is scalar (not funct. of } n) \\ &= \sum_{k=-\infty}^{\infty} x[k]h_k[n] \end{aligned}$$

where we have used the linearity property in the 3rd equality.

Theorem: Every DT LSI system can be completely characterized by its impulse response, that is the response (output) of the system to the unit-pulse sequence.

Proof of Theorem: Since the linear system is also SI, if $h_0[n] = h[n]$ is the response of the system to $\delta[n]$ then $h[n - k]$ is the response of the system to $\delta[n - k]$. The results of the above lemma then become:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h_k[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= x[n] * h[n] \end{aligned}$$

where * denotes the convolution operator.

As a consequence of the above theorem, a linear, SI system is completely characterized by its impulse response, $h[n]$, in the sense that given $h[n]$ and any input sequence $\{x[n]\}$, the output of the system, $\{y[n]\}$ can be computed from the convolution of the sequences, $y[n] = x[n] * h[n]$.

Example: (Convolution obtained graphically)

$x[0] = 0.5$, $x[1] = 0.5$, $x[2] = 0.5$
$h[0] = 3$, $h[1] = 2$, $h[2] = 1$

Example: [Convolution using vector inner (dot) product give a finite-length impulse response (FIR)]

$$y[0] = [h[0] \quad 0 \quad 0]\begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = h[0]x[0]$$

$$y[1] = [h[1] \quad h[0] \quad 0]\begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = h[1]x[0] + h[0]x[1]$$

$$y[2] = [h[2] \quad h[1] \quad h[0]]\begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = h[2]x[0] + h[1]x[1] + h[0]x[2]$$

$$y[3] = [0 \quad h[2] \quad h[1]]\begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = h[2]x[1] + h[1]x[2]$$

$$y[4] = [0 \quad 0 \quad h[2]]\begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = h[2]x[2]$$

In this example, we shift the coefficients of the impulse response [first-in, first out (FIFO)]. Note that the primary operation here is multiply and add (accumulate) (MAC). The MAC instruction on a digital signal processor (DSP) is one difference between a DSP and general-purpose microprocessors. Typically MACs occur in one clock cycle and allow simultaneous memory move(s).

From the convolution formula, we see for causal LSI systems, we must have

$$h[n] = 0, \; n < 0.$$

Properties of Linear, Shift-Invariant (LSI) Systems

The identity sequence for the convolution operator is $\delta[n]$: $x[n] * \delta[n] = x[n]$

Properties of the convolution operator:

1) Commutative: $x[n] * y[n] = y[n] * x[n]$

We note this commutative property is valid only for real-valued systems and inputs. In the complex-valued case

$$\begin{aligned} y(n) &= h^H(n)x(n) \\ &\neq x^H(n)h(n) \end{aligned}$$

where ^{H} is the Hermitian transpose (conjugate transpose).

2) Associative: $x[n] * \{y[n] * z[n]\} = \{x[n] * y[n]\} * z[n]$

3) Distributive: $x[n] * \{y[n] + z[n]\} = x[n] * y[n] + x[n] * z[n]$

Theorem: A linear, SI system with impulse response, $h(n)$ is BIBO stable iff $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$ (*P* iff *Q*). For absolutely summable sequences, we will also use the notation $\mathbf{h} \in l_1$ or $\|\mathbf{h}\|_1 < \infty$.

Proof: [Sufficient ($Q \Rightarrow P$)] - assume $S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$. Show for every **x** bounded in, **y** is bounded out.)

$$
\begin{aligned}
|y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\
&\le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \quad ;\text{triangle inequality } |a+b| \le |a| + |b| \\
&\le \sum_{k=-\infty}^{\infty} |h[k]| B_x \quad ;\text{assumption that } \mathbf{x} \text{ is bounded} \\
&= B_x \sum_{k=-\infty}^{\infty} |h[k]| \quad ;\text{assume S} = \sum_{k=-\infty}^{\infty} |h[k]| \\
&= B_y
\end{aligned}
$$

Thus **y** is bounded out.

[Necessary ($P \Rightarrow Q$ or equivalently $\sim Q \Rightarrow \sim P$] - assume BIBO, show $S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$ or conversely, assume $S = \infty$, show not BIBO, i.e. a bounded input leads to an unbounded output.)

Consider

$$
x[n] = \begin{cases} \dfrac{h^*[-n]}{|h[-n]|} & , \ h[n] \ne 0 \\ 0 & , \ h[n] = 0 \end{cases}
$$

Clearly, $|x[n]| = 1$ (bounded in) but

$$
\begin{aligned}
y[0] &= \sum_{k=-\infty}^{\infty} x[-k]h[k]; \quad y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] \text{ (Convolution)} \\
&= \sum_{k=-\infty}^{\infty} \frac{h[k]^* h[k]}{|h[k]|} \\
&= \sum_{k=-\infty}^{\infty} \frac{|h[k]|^2}{|h[k]|} \\
&= \sum_{k=-\infty}^{\infty} |h[k]| \\
&= S \\
&= \infty
\end{aligned}
$$

Clearly, FIR systems will always be stable.