Discrete-Time Signal and Systems

Discrete-TIme Signals: Sequences

Definition: A **DT signal** is a sequence, that is a function defined on the positive and negative integers.

The n^{th} number (element) of a sequence (column vector) **x** is denoted, $x[n]$.

Definition: A DT signal whose amplitude values are from a finite set is called a **digital signal**.

Typically such sequences arise from periodic sampling of an analog signal. In this case, the numeric value of the n^{th} number in the sequence is equal to the value (measurement) of the analog signal $x_a(t)$ at time nT

$$x[n] = x_a(nT).$$

Definition: A sequence $y[n]$ is said to be a shifted version of a sequence $x[n]$ if $y[n]$ has values

$$y[n] = x[n - n_0]$$

for n_0 an integer. If n_0 is positive, $y[n]$ is a delayed version of $x[n]$.

Figure: Signal with $n_0 = 3$.

Definition: The unit sample or unit-pulse or impulse sequence, $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Any sequence can be represented by a sum of scaled, shifted unit-pulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Example: (trivial example of above similar to (2.5))

Definition: The unit-step sequence, $u[n]$ is defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}.$$

Definition: The exponential sequence has the form

$$x[n] = A\alpha^n.$$

If A and α are real, $x[n]$ is real. If $|\alpha| > 1$, $|\alpha| < 1$ the sequence increases, decreases respectively in magnitude as n increases.

Definition: The complex exponential sequence ($|\alpha| = 1$) has the form

$$\begin{aligned} x[n] &= |A| e^{j(\omega_0 n + \varphi)} \\ &= |A| \cos(\omega_0 n + \varphi) + j|A| \sin(\omega_0 n + \varphi) \end{aligned}.$$

Definition: The sinusoidal sequence has the form

$$x[n] = A\cos(\omega_0 n + \varphi)$$

with *A* real.

We make a few observations about DT sinusoidal and complex exponential sequences.

⇒ 1) Consider complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$ where *r* is an integer. These sequences are indistinguishable from one another

$$\begin{aligned} x[n] &= Ae^{j(\omega_0 + 2\pi r)n} \\ &= Ae^{j\omega_0 n} \underbrace{e^{j2\pi rn}}_{=1} \\ &= Ae^{j\omega_0 n} \end{aligned}.$$

We therefore consider frequencies only in the range $0 \le \omega_0 < 2\pi$. There will be implications of this when we sample signals.

⇒ 2) Consider a CT complex exponential with frequency $\Omega = 2\pi f$ (rads/s). Clearly $\Omega T = 2\pi f T = \frac{2\pi}{T} T = 2\pi$. CT sinusoidal and complex exponential signals are *always* periodic with period $T = 2\pi / \Omega$

$$\begin{aligned} x(t+T) &= e^{j\Omega(t+T)} \\ &= e^{j\Omega t} e^{j\Omega T} \\ &= e^{j\Omega t} \\ &= x(t) \end{aligned}.$$

Now, a periodic DT sequence (with period *N*) satisfies

$$x[n+N] = x[n].$$

Consider a DT complex exponential sequence with frequency ω_0 in rads/sample. Periodicity with period *N* requires

$$Ae^{j\omega_0(n+N)} = Ae^{j\omega_0 n}$$

which is true *only* when $\omega_0 N = 2\pi r$ or $\omega_0 = 2\pi \frac{r}{N}$ (i.e. a rational number times 2π). Consequently, complex exponential (and sinusoidal) sequences are not necessarily periodic with period $2\pi / \omega_0$.

⇒ 3) For a CT sinusoid

$$x(t) = A\cos(\Omega_0 t + \varphi)$$

as Ω_0 increases, $x(t)$ oscillates more and more rapidly. For a DT sinusoid

$$x[n] = A\cos(\omega_0 n + \varphi)$$

for $0 \le \omega_0 \le \pi$, as ω_0 increases, $x[n]$ oscillates more and more rapidly. However, for $\pi \le \omega_0 \le 2\pi$, as ω_0 increases, the oscillations become slower.