Determinants

Introduction to the "Easy Problems"

We want conditions on the solution to the system of linear equations [A_{ij} (ith row, jth column), b_k (kth element) are known; x_k is unknown]

$$A_{11}x_1 + \cdots + A_{1n}x_n = b_1$$
$$\vdots$$
$$A_{n1}x_1 + \cdots + A_{nn}x_n = b_n$$

or more compactly,

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \cdots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$
$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

The conditions on the solution to this "easy" problem are:

if det($\mathbf{A}$) = 0 there is no solution or infinitely many solutions
if det($\mathbf{A}$) ≠ 0 there is exactly one solution.

Historically, the "easy" problems were studied first and are very old.

Introduction to the "Hard Problems"

A second category of problems might be called the "spectral theory" of matrices and involves such things as eigenvalues, eigenvectors, and functions of a matrix i.e. if $f(x)$ is a polynomial in x with coefficients f_k, then

$$f(\alpha) = f_0 + f_1\alpha + f_1\alpha^2 + \cdots + f_n\alpha^n \text{ ,(}\alpha \text{ scalar)}$$
$$f(\mathbf{A}) = f_0 + f_1\mathbf{A} + f_1\mathbf{A}^2 + \cdots + f_n\mathbf{A}^n \text{ ,(}\mathbf{A} \in C^{m\times m}\text{ , matrix)}$$

Example Some functions of a matrix which are of particular interest include $e^{\mathbf{A}t}$ and $\mathbf{A}^t$.

The theory of determinants leads to the Cayley-Hamilton theory which in turn leads to the Spectral Theory and Functional Calculus. The complete progression to the Spectral Theory is

Determinants → Cramer's Rule → Cayley-Hamilton Theorem → Spectral Theory
(easy) (bridge) (hard)

Theorem (*Cramer's Rule*) Let **C** be a matrix of cofactors with

$$C_{ij} = \frac{\partial}{\partial A_{ij}} \det(\mathbf{A})$$

Then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T.$$

Theorem (*Cayley-Hamilton*) Let $p(s) = \det(s\mathbf{I} - \mathbf{A})$ be the characteristic polynomial. Then $p(\mathbf{A}) = \mathbf{0}$.

The spectral theory was not studied until the 19th century. We say that a matrix **A** is *normal* if $\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}$ where * denotes conjugate transpose. We say that a matrix is *unitary* if $\mathbf{A}^* = \mathbf{A}^{-1}$. We are already familiar with a limited version of the spectral theory.

Theorem (*Spectral Theorem for Normal Matrices*) Every normal matrix A is unitarily equivalent to a diagonal matrix (whose diagonal elements must be the eigenvalues of **A**). In other words, there exist a unitary matrix **T** and diagonal matrix **D** for which

$$\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^* .$$

where

$$\mathbf{T} = \left[\nu_1 \mid \nu_2 \mid \cdots \mid \nu_n \right],$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

and ν_k is an eigenvector with λ_k the associated eigenvalue.

Notation

C is the set of complex numbers
C^n is the set of complex vectors of size n
$C^{n\times n}$ is the set of $n \times n$ complex matrices
$\mathbf{A} = \left[\mathbf{A}^{(1)}, \quad \cdots \quad , \mathbf{A}^{(n)} \right] \in C^{n\times n}$ having columns $\mathbf{A}^{(k)} \in C^n$

What is det?

A great number of mathematical questions can be answered by computing determinants:

- Does there exist a solution to a system of linear equations?
- When does a polynomial have no roots in the right half plane or outside the unit circle? Answers the question of causality and stability of CT and DT LTI systems.
- When is a matrix positive definite, i.e. $\mathbf{x}^T \mathbf{A}\mathbf{x} > 0$ for all non-zero vectors, **x**?
- Is a given pair of polynomials co-prime?
- How far is a vector from a subspace?
- What is the volume of a skewed parallelogram?

Towards a Definition of Determinant

Q: Does there exist a scalar function $\det : C^{n\times n} \mapsto C$ with the following properties?

1) det is linear in each column, i.e.

$$\det\left[\cdots \mid a\mathbf{x} + b\mathbf{y} \mid \cdots \right] = a \det\left[\cdots \mid \mathbf{x} \mid \cdots \right] + b \det\left[\cdots \mid \mathbf{y} \mid \cdots \right]$$

2) if $\mathbf{A}^{(k)} = \mathbf{A}^{(j)}$ (kth column of **A**, jth column of **A**) then $\det(\mathbf{A}) = 0$
3) $\det(\mathbf{I}) = 1$ where the *identity* matrix is given by

$$\mathbf{I} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

As we will see, these three properties completely and uniquely specify the determinant. Consider some consequences of these properties without really knowing whether “det” actually exists.

Example 4) If the matrix **A** has a zero column, then the matrix has zero determinant:

$$\mathbf{A}^{(k)} = \mathbf{0} \Rightarrow \det(\mathbf{A}) = 0.$$

This follows from Property 1:

$$\det[\cdots \mid a\mathbf{x} \mid \cdots] = a \det[\cdots \mid \mathbf{x} \mid \cdots]$$

with $a = 0$.

Example 5) If a multiple of one column is added to another, the determinant is not changed. This follows from Properties 1 and 2. Operating on column j:

$$\begin{aligned}\det\left[\cdots \mid \mathbf{A}^{(j)} + \alpha\mathbf{A}^{(k)} \mid \cdots\right] &= \det\left[\cdots \mid \mathbf{A}^{(j)} \mid \cdots\right] + \alpha \det\left[\cdots \mid \mathbf{A}^{(k)} \mid \cdots\right] \\ &= \det(\mathbf{A}) + 0\end{aligned}$$

We will show that if det(**A**) ≠ 0, then **A** has an inverse, namely $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. If this $\mathbf{A}^{-1}$ exists then the solution to $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Primitive Matrices

A matrix **T** is called *primitive* if it differs from the identity matrix in at most one position. Thus

$$T_{ij} = \begin{cases} \alpha, & i = u, j = v \\ \delta_{ij}, & otherwise \end{cases}$$

where

$$\delta_{ij} \equiv \begin{cases} 1, & i = j \\ 0, & otherwise \end{cases}.$$

T is parameterized by the triple (u, v, α).

Examples: $\mathbf{T}(u,v,\alpha) = \begin{bmatrix} 1 & & \\ & \ddots & \alpha \\ & & 1 \end{bmatrix}$, $\mathbf{T}(u,v,\alpha) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \alpha & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$, $\mathbf{T}(2,3,1.7) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1.7 \\ 0 & 0 & 1 \end{bmatrix}$.

Exercise 1: Establish the following

$$\det(\mathbf{A}\,\mathbf{T}) = \det(\mathbf{A})\det(\mathbf{T})$$

if **T** is primitive.