New Mexico State University
Klipsch School of Electrical Engineering

EE555 - Advanced Linear Systems
Spring 2005
Exam #2

Name: ____________________

Solve four problems–two from each half of exam.

Prob. 1	/ 25 points
Prob. 2	/ 25 points
Prob. 3	/ 25 points
Prob. 4	/ 25 points
Prob. 5	/ 25 points
Prob. 6	/ 25 points
Total	/ 100 points

Prob. 1

(i) True or False: If $\mathbf{B} = \mathbf{U}^*\mathbf{A}\mathbf{U}$ is normal with $\mathbf{U}$ unitary, then $\mathbf{A}$ must be normal. If true you must prove so, otherwise simply write false (no counter-example required).

(ii) (Text p. 70) Prove that if $\mathbf{X} \in \mathbb{C}^{n\times m}$ and $\mathbf{X}^*\mathbf{X} = \mathbf{I}$, then there exists a matrix $\mathbf{Y} \in \mathbb{C}^{n\times(n-m)}$ such that $[\mathbf{X}, \mathbf{Y}]$ is unitary. Assume $m < n$.

(iii) [Text Prob. IX 23(ii) p. 142] Prove that if $\mathbf{Y} > \mathbf{0}$ and $\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \mathbf{X}^* & \mathbf{0} \end{bmatrix} \geq \mathbf{0} \implies \mathbf{X} = \mathbf{0}$.

Prob. 1 (cont.)

Prob. 2

Prove the following:

(i) (Text p. 72) $\mathbf{A}$ Hermitian $\iff$ $\mathbf{A}$ normal and $\lambda_k^* = \lambda_k$ for each k.

(ii) (Text p. 72) $\mathbf{A}$ unitary $\iff$ $\mathbf{A}$ normal and $|\lambda_k| = 1$ for each k.

(iii) Prove that if $\mathbf{A} - \mathbf{B} \geq \mathbf{0}$, then the diagonal elements of $\mathbf{A} - \mathbf{B}$ must be non-negative.

Prob. 2 (cont.)

Prob. 3

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(i) Find $\mathbf{x} \in \text{Range}\,(\mathbf{A})$ which is nearest $\mathbf{y}$.

(ii) Find $\mathbf{x} \in \text{Null}\,(\mathbf{A})^{\perp}$ which is nearest $\mathbf{y}$.

Prob. 3 (cont.)

Prob. 4

Prove the following:

(i) $\text{Null}(\mathbf{C}) \subset \text{Null}(\mathbf{B})$ if and only if $\mathbf{B}(\mathbf{I} - \mathbf{C}^{+}\mathbf{C}) = \mathbf{0}$.

(ii) Suppose that $\mathbf{A}$ is square and $\mathbf{C} = \mathbf{C}^{*}$. Then $\text{Null}(\mathbf{C}) \subset \text{Null}(\mathbf{B})$ if and only if

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{*} & \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{B}\mathbf{C}^{+} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{C}^{+}\mathbf{B}^{*} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{C}^{+})^{*}\mathbf{B}^{*} & \mathbf{I} \end{bmatrix}$$

Note: The corollary to (i) is $\text{Null}(\mathbf{C}) \subset \text{Null}(\mathbf{B})$ if and only if $(\mathbf{I} - \mathbf{C}\mathbf{C}^{+})\mathbf{B}^{*} = \mathbf{0}$. You may wish to use (i) and the corollary to prove (ii).

Prob. 4 (cont.)

Prob. 5

Prove the following:

(i) $\text{Null}(\mathbf{A}) \subset \text{Null}(\mathbf{B}^*)$ if and only if $\mathbf{B}^*(\mathbf{I} - \mathbf{A}^+\mathbf{A}) = \mathbf{0}$.

(ii) Assuming $\mathbf{B}^*(\mathbf{I} - \mathbf{A}^+\mathbf{A}) = \mathbf{0}$ and $(\mathbf{I} - \mathbf{A}\mathbf{A}^+)\mathbf{B} = \mathbf{0}$, verify the factorization

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B}^*\mathbf{A}^+ & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} - \mathbf{B}^*\mathbf{A}^+\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}^+\mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

(iii) [Text Prob. IX 15(i) p. 140] Let $\mathbf{Q} = \mathbf{Q}^* = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{bmatrix}$. Prove that

$$\mathbf{Q} \geq \mathbf{0} \iff \mathbf{A} \geq \mathbf{0} \text{ and } \mathbf{D} \geq \mathbf{B}^*\mathbf{A}^+\mathbf{B} \text{ and } \text{Null}(\mathbf{A}) \subset \text{Null}(\mathbf{B}^*).$$

Note: The corollary to (i) is $\text{Null}(\mathbf{A}) \subset \text{Null}(\mathbf{B}^*)$ if and only if $(\mathbf{I} - \mathbf{A}\mathbf{A}^+)\mathbf{B} = \mathbf{0}$. You may wish to use (i), (ii), and the corollary in your proof of (iii).

Prob. 5 (cont.)

Prob. 6

Consider the *weighted* least-squares (LS) problem:

- Given $\mathbf{A} \in \mathbb{C}^{n \times m}$, $\mathbf{y} \in \mathbb{C}^n$, and the "weighting matrix" $\mathbf{W} > \mathbf{0}$
- Minimize $V(\mathbf{x}) = (\mathbf{y} - \mathbf{A}\mathbf{x})^* \mathbf{W} (\mathbf{y} - \mathbf{A}\mathbf{x}) = ||\mathbf{y} - \mathbf{A}\mathbf{x}||_W^2$

(i) Determine the optimal solution, $\mathbf{x}_{\text{opt}}$ assuming $n \geq m$ and the weighted equivalent of $\mathbf{A}^*\mathbf{A}$ (for the conventional LS problem) is non-singular. The optimal solution is the one for which $V(\mathbf{x}) \geq V(\mathbf{x}_{\text{opt}})$.

(ii) Determine the optimal solution, $\mathbf{x}_{\text{opt}}$ assuming $n \geq m$ and the weighted equivalent of $\mathbf{A}^*\mathbf{A}$ (for the conventional LS problem) is singular.

Note: In the conventional LS problem (overdetermined) of minimizing $V(\mathbf{x}) = ||\mathbf{y} - \mathbf{A}\mathbf{x}||^2$ we had two cases of interest: one where $\mathbf{A}^*\mathbf{A}$ is non-singular and one where $\mathbf{A}^*\mathbf{A}$ is singular. The latter led to the use of the pseudoinverse.

Prob. 6 (cont.)