Eigenfilters

Consider the following

Figure 4.4

We assume

- filter is FIR, i.e. $\mathbf{w} = [\,\omega_0 \quad \Lambda \quad \omega_{M-1}]^T$
- *u*(*n*) is a random signal (WSS) with zero-mean and correlation matrix **R**
- *v*(*n*) is a zero-mean, white noise signal with variance σ^2
- signal and noise are uncorrelated, i.e. $E[u(n)\overset{*}{\varpi}(\mu)] = 0$ for all *m* and *n*.

We wish to design **w** so that the signal-to-noise ratio (SNR) is maximized at the output of the filter. This is the stochastic version of the matched filter which maximizes the output SNR for a known signal in additive noise.

Since the filter is linear we may consider the signal and noise power separately. You will show (Prob. 2.9) that the average output power due to the signal is given by

$$P_o = \boldsymbol{\omega}^H \mathbf{P} \boldsymbol{\omega}$$

and the average output power due to the noise is given by

$$N_o = \sigma^2 \boldsymbol{\omega}^H \boldsymbol{\omega} \,.$$

We therefore wish to maximize the output SNR

$$\begin{aligned}(SNR)_0 &= P_0 / N_0 \\ &= \frac{\mathbf{w}^H \mathbf{R} \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}} \\ &= \frac{1}{\sigma^2} \frac{\mathbf{w}^H \mathbf{R} \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}}\end{aligned}$$

subject to the constraint that the filter has unit noise gain

$$\mathbf{w}^H \mathbf{w} = 1 \,.$$

Theorem: The maximum output SNR is

$$(SNR)_{o,\,\mu\alpha\xi} = \frac{\lambda_{\mu\alpha\xi}}{\sigma^2} \,.$$

Proof: Trivial by the minimax theorem.

Theorem: The coefficient vector of the optimum FIR filter $\mathbf{w}_{\text{opt}}$ which achieves maximum output SNR is defined by

$$\mathbf{w}_{\text{opt}} = \boldsymbol{\theta}_{\mu\alpha\xi}$$

where $\mathbf{q}_{\max}$ is the eigenvector associated with $\lambda_{\max}$. We call this filter an *eigenfilter*.

Proof: Let

$$\mathbf{w} = \boldsymbol{\Theta}\boldsymbol{\omega}'$$

where **Q** is the matrix that diagonalizes **R**, i.e. $\mathbf{Q}^H\mathbf{R}\mathbf{Q} = \Lambda$. Note that

$$\mathbf{w}' = \mathbf{Q}^{-1}\mathbf{w} = \mathbf{Q}^H\mathbf{w}$$

thus

$$\begin{aligned}\mathbf{w}'^H\mathbf{w}' &= \mathbf{w}^H\mathbf{Q}\mathbf{Q}^H\mathbf{w}\\ &= \mathbf{w}^H\mathbf{w}\\ &= 1\end{aligned}$$

since **Q** is unitary. Then

$$\begin{aligned}\frac{\mathbf{w}^H\mathbf{R}\mathbf{w}}{\mathbf{w}^H\mathbf{w}} &= \frac{\mathbf{w}'^H\mathbf{Q}^H\mathbf{R}\mathbf{Q}\mathbf{w}'}{\mathbf{w}'^H\mathbf{Q}^H\mathbf{Q}\mathbf{w}'}\\ &= \frac{\mathbf{w}'^H\Lambda\mathbf{w}'}{\mathbf{w}'^H\mathbf{w}'}\\ &= \frac{\sum_{i=1}^{M}\lambda_i w_i'^* w_i'}{\mathbf{w}'^H\mathbf{w}'} \qquad ; \mathbf{w}'^H\mathbf{w}' = 1\\ &= \sum_{i=1}^{M}\lambda_i w_i'^* w_i'\end{aligned}$$

We can maximize this quantity by choosing

$$w_i' = \begin{cases}1, & i = l\\ 0, & i \neq l\end{cases}$$

where $\lambda_l = \lambda_{\max}$ is the largest eigenvalue, i.e. given the unit norm constraint, maximize the sum by weighting the largest eigenvalue by one and zeroing out all the other eigenvalues. Then the eigenfilter is given by

$$\begin{aligned}\mathbf{w} &= \boldsymbol{\Theta}\boldsymbol{\omega}'\\ &= [\boldsymbol{\theta}_1 \ | \Lambda \ | \boldsymbol{\theta}_\lambda \ | \Lambda \ | \boldsymbol{\theta}_M]\begin{bmatrix}0\\ \mathrm{M}\\ 0\\ 1\\ 0\\ \mathrm{M}\\ 0\end{bmatrix}\\ &= \boldsymbol{\theta}_\lambda\end{aligned}$$

which is equivalent to the eigenvector associated with the largest eigenvalue.