Spectrum Analysis

Power Spectral Density

Autocorrelation: time-domain description of second-order statistics of a stochastic process.

Power Spectral Density (PSD or power spectrum of spectrum): frequency-domain description of second-order statistics of a stochastic process

PSD describes how power is expected to be distributed in the spectrum. We have the PSD of a WSS process defined as

$$S(\omega) = \sum_{l=-\infty}^{\infty} r(l) e^{-j\omega l}, -\pi \le \omega \le \pi$$

where *r*(*l*) is the autocorrelation of the process of interest. We note that *S*(ω) is 2π periodic and if *r*(*l*) is real-valued than *S*(ω) is conjugate symmetric about π.

Properties of the PSD

Property 1: The autocorrelation function and PSD form a DTFT pair (Einstein-Wiener-Khintchine relations)

$$r(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega) e^{j\omega l} d\omega \overset{DTFT}{\leftrightarrow} S(\omega) = \sum_{l=-\infty}^{\infty} r(l) e^{-j\omega l}$$

with $l = 0, \pm 1, \pm 2, \ldots$.

Property 2: The PSD is a real-valued function in ω. We have

$$\begin{aligned} S(\omega) &= \sum_{l=-\infty}^{\infty} r(l) e^{-j\omega l} \\ &= r(0) + \sum_{l=1}^{\infty} r(l) e^{-j\omega l} + \sum_{l=-\infty}^{-1} r(l) e^{-j\omega l} \\ &= r(0) + \sum_{l=1}^{\infty} r(l) e^{-j\omega l} + \sum_{l=1}^{\infty} r(-l) e^{j\omega l} \\ &= r(0) + \sum_{l=1}^{\infty} [r(l) e^{-j\omega l} + r^*(k) e^{j\omega l}], \quad r(-k) = r^*(k) \\ &= r(0) + 2\sum_{l=1}^{\infty} \mathrm{Re}[r(l) e^{-j\omega l}] \end{aligned}$$

where for a zero-mean process, $r(0) = E[u(n)u^*(n)] = \sigma^2$ is the variance.

Property 3: For *l* = 0 we have *r*(*0*) or the mean-square value of the process or the expected power of the process

$$r(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega) d\omega \ .$$

This validates the use of the term power spectral density for $S(\omega)$.

Linear Filtering of a Stationary Process

Figure 3.1

The output of the filter is given by the convolution

$$y(n) = \sum_{i=-\infty}^{\infty} h(i)u(n-i)$$

and likewise

$$y^*(n-l) = \sum_{k=-\infty}^{\infty} h^*(k)u^*(n-l-k)\,.$$

Taking the expectation of the product of the above outputs we have

$$\begin{aligned}\underbrace{E[y(n)y^*(n-l)]}_{r_y(l)} &= E\left[\sum_{i=-\infty}^{\infty} h(i)u(n-i)\sum_{k=-\infty}^{\infty} h^*(k)u^*(n-l-k)\right]\\ &= \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h(i)h^*(k)\underbrace{E[u(n-i)u^*(n-l-k)]}_{r_u(k-i+l)}\end{aligned}\,.$$

Taking the DTFT of the above we have

$$\begin{aligned}\sum_{l=-\infty}^{\infty} r_y(l)e^{-j\omega l} &= \sum_{l=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h(i)h^*(k)r_u(k-i+l)e^{-j\omega l}\\ &= \sum_{l=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h(i)h^*(k)r_u(k-i+l)e^{-j\omega i}e^{j\omega k}e^{-j\omega(k-i+l)}\\ &= \underbrace{\sum_{i=-\infty}^{\infty} h(i)e^{-j\omega i}}_{H(e^{j\omega})}\underbrace{\sum_{k=-\infty}^{\infty} h^*(k)e^{j\omega k}}_{H^*(e^{j\omega})}\underbrace{\sum_{l=-\infty}^{\infty} r_u(k-i+l)e^{-j\omega(k-i+l)}}_{S_u(\omega)}\,.\end{aligned}$$

$$\Leftrightarrow$$

$$S_y(\omega) = \left|H(e^{j\omega})\right|^2 S_u(\omega)$$

We therefore have that the output PSD of a filtered process is equal to the input PSD of the process times the magnitude-squared response of the filter.