Stationary Processes and Models

Stochastic Models

Figure 2.1

An autoregressive moving average (ARMA) process, *u*(*n*) results from the output of a system

$$u(n) = \sum_{k=0}^{K} b_k^* v(n-k) - \sum_{k=1}^{M} a_k^* u(n-k)$$

whose input, *v*(*n*) is a white, usually Gaussian, noise. We have

$$E[v(n)] = 0$$

$$E[v(n)v^*(k)] = \begin{cases} \sigma_v^2, & k = n \\ 0, & o.w. \end{cases}$$

where σ_v^2 is the input noise variance. We call the a_k's the AR coefficients and the b_k's the MA coefficients.

It turns out many stochastic processes which occur in engineering and science can be modeled as ARMA processes (white noise through a linear filter).

Autoregressive Models

If $b_1, \ldots, b_K = 0$, the resulting process is called an autoregressive (AR) model. As a process generator or synthesizer, [meaning that given the white noise *v*(*n*) we produce the process *u*(*n*)] this model is all-pole.

Given the correlations of an AR process, it is simple to determine the AR coefficients of the model (Yule-Walker).

Moving Average Models

If $a_1, \ldots, a_M = 0$ the resulting process is called an moving average (MA) model. As a process synthesizer this model is all-zero.

Given the correlations of an MA process, it is not simple to determine the MA coefficients of the model.

Wold Decomposition

The wide application of AR processes is justified by the Wold decomposition theorem.

Theorem: Any stationary DT process, *x*(*n*) can be decomposed as

$$x(n) = u(n) + s(n)$$

where

1. *u*(*n*) and *s*(*n*) are uncorrelated processes, i.e. $E[u(n)s^*(k)] = 0$ for all n, k;

2. *u*(*n*) is a general linear processes represented by the MA model,

$$u(n) = \sum_{k=0}^{\infty} b_k^* v(n-k)$$

with $b_0 = 1$, and

$$\sum_{k=0}^{\infty} |b_k|^2 < \infty,$$

where $v(n)$ is white noise uncorrelated with $s(n)$;

3. $s(n)$ is a predictable process; that is, the process can be predicted from its own past with zero prediction variance.

Yule-Walker Equations

Given the correlations of a process

$$r(l) = E[u(n)u^*(n-l)]$$

we wish to determine the coefficients of an AR model assuming a model order of M. We have the AR process modeled as

$$u(n) = v(n) - \sum_{k=1}^{M} a_k^* u(n-k)$$

or equivalently,

$$\sum_{k=0}^{M} a_k^* u(n-k) = v(n)$$

where we assume $a_0^* = 1$ and $v(n)$ is white noise. As a first step we towards $r(l)$ we have

$$E[v(n)u^*(n-l)] = E\left[\sum_{k=0}^{M} a_k^* u(n-k)u^*(n-l)\right]$$
$$= \sum_{k=0}^{M} a_k^* E[u(n-k)u^*(n-l)] \; .$$
$$= \sum_{k=0}^{M} a_k^* r(l-k)$$

Figure 2.2: AR process synthesizer

We note that

$$E[v(n)u^*(n-l)] = 0, \; l > 0$$

since $u(n-l)$ only involves inputs up to $n-l$, i.e. $v(n-l)$ all of which are uncorrelated with $v(n)$. We therefore have

$$\sum_{k=0}^{M} a_k^* r(l-k) = 0, \; l > 0$$

or

$$r(l) = -\sum_{\kappa=1}^{M} \alpha_{\kappa}^{*} \rho(\lambda - \kappa), \ \lambda > 0$$
$$= \sum_{\kappa=1}^{M} \omega_{\kappa}^{*} \rho(\lambda - \kappa)$$

where $w_k = -\alpha_{\kappa}$. We can write a system of equations using the conjugate of the above for $l = 1, 2, \ldots, M$ and express in matrix form as (Yule-Walker equations)

$$\underbrace{\begin{bmatrix} r(0) & \rho(1) & \cdots & \rho(M-1) \\ \rho^{*}(1) & \rho(0) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{*}(M-1) & \rho^{*}(M-2) & \cdots & \rho(0) \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_M \end{bmatrix}}_{\boldsymbol{\omega}} = \underbrace{\begin{bmatrix} \rho^{*}(1) \\ \rho^{*}(2) \\ \vdots \\ \rho^{*}(M) \end{bmatrix}}_{\boldsymbol{\rho}}.$$

Assuming **R** is nonsingular, we have as a solution to the Yule-Walker equations

$$\mathbf{w} = \mathbf{P}^{-1}\boldsymbol{\rho}.$$

Thus given the correlations of a process we can determine the coefficients of an AR model of order of *M*.