State-Space Representations and Examples

In the usual study of DSP, we describe LTI systems with an impulse response, $h(n)$; frequency response, $H(e^{j\omega})$; difference equation, $y(n) = -\sum_{m=1}^{M} a_m y(n-m) + \sum_{l=0}^{L} b_l x(n-l)$; transfer function, $H(z)$; or a pole/zero pattern.

We also can describe a linear system with a state-variable description or state-space representation. Consider the following example.

Example: Suppose we have a simple second order difference equation

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + x(n) + b_1 x(n-1) + b_2 x(n-2)$$

and the equivalent Direct Form II (Canonical) realization.

Figure (actual DFII and generalized "state" structure)

To be more general, we denote the internal filter "state" with S and the state variables with $q_i(n)$. The block diagram indicates that the next value of the state is obtained from the current state and the input via the equations

$$\begin{aligned} q_1(n) &= -a_1 q_1(n-1) - a_2 q_2(n-1) + x(n) \\ q_2(n) &= q_1(n-1) \end{aligned}$$

The block diagram also indicates that the system output is expressed in terms of the input and state as

$$y(n) = (b_1 - a_1) q_1(n) + (b_2 - a_2) q_2(n) + x(n).$$

In matrix form we can write a state update equation as

$$\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(n-1) \\ q_2(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(n)$$

$$\mathbf{q}(n) = \mathbf{A}\mathbf{q}(n-1) + \mathbf{b}x(n)$$

and the output equation as

$$y(n) = [(b_1 - a_1) \quad (b_2 - a_2)] \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + [1]x(n)$$

$$y(n) = \mathbf{c}\mathbf{q}(n) + dx(n)$$

Using the standard notation for state-space descriptions we can generalize for multiple inputs and multiple outputs (MIMO) and allow for the linear system to be dynamical (time-varying)

$$\boxed{\begin{aligned} \mathbf{x}(n) &= \mathbf{A}(n-1)\mathbf{x}(n-1) + \mathbf{B}(n)\mathbf{w}(n) \\ \mathbf{y}(n) &= \mathbf{C}(n)\mathbf{x}(n) + \mathbf{D}(n)\mathbf{v}(n) \end{aligned}}$$

where

State vector or signal model: $\mathbf{x}(n) = [x_1(n) \quad x_2(n) \quad \cdots \quad x_p(n)]^T$

Output vector or observation model or measurement model: $\mathbf{y}(n) = [y_1(n) \quad y_2(n) \quad \cdots \quad y_q(n)]^T$

Process noise vector or modeling error vector: $\mathbf{w}(n) = [w_1(n) \quad w_2(n) \quad \cdots \quad w_p(n)]^T$

Measurement noise vector or observation error vector: $\mathbf{v}(n) = [v_1(n) \quad v_2(n) \quad \cdots \quad v_q(n)]^T$

State transition matrix, $\mathbf{A}(n)$ ($p \times p$)

Input matrix, $\mathbf{B}(n)$ ($\times\, p$)

Observation matrix or measurement matrix or output matrix, $\mathbf{C}(n)$ ($q \times p$)

Direct transmission matrix, $\mathbf{D}(n)$ ($\times\, q$)

Note that with this new notation, $\mathbf{w}(n)$ is the input to the signal model and $\mathbf{x}(n)$ is the input to the observation model

Figure 7.11 (Manolakis)

It should be noted that from the state-space representation we can get alternate descriptions [**D** a scalar (straight-through path] such as the transfer function,

$$H(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

or the impulse response (for causal systems)

$$h(n) = \begin{cases} 0, & n < 0 \\ \mathbf{D}, & k = 0 \\ \mathbf{C}\mathbf{A}^{n-1}\mathbf{B}, & k > 0 \end{cases}.$$

or the poles of the system. We have $A(z) = \det(z\mathbf{I} - \mathbf{A})$ thus the eigenvalues of **A** are the poles. Similar result for zeros.

Example (Manolakis): Suppose we have an AR process described by (not the usual notation from Chapter 2)

$$x(n) = 1.8x(n-1) - 0.81x(n-2) + 0.1w(n)$$

where $w(n)$ is zero mean, unit variance white Gaussian noise. Suppose we can only make "noisy" measurements of the process

$$y(n) = x(n) + \sqrt{10}v(n)$$

where $v(n)$ is zero mean, variance of 10 white Gaussian noise independent of $w(n)$. We have as our state-space description

$$\boxed{\begin{aligned} \mathbf{x}(n) &= \mathbf{A}(n-1)\mathbf{x}(n-1) + \mathbf{B}(n)\mathbf{w}(n) \\ \mathbf{y}(n) &= \mathbf{C}(n)\mathbf{x}(n) + \mathbf{D}(n)\mathbf{v}(n) \end{aligned}}$$

where

$$\mathbf{A}(n) = \begin{bmatrix} 1.8 & -0.81 \\ 1 & 0 \end{bmatrix}, \mathbf{B}(n) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \mathbf{C}(n) = [1 \quad 0], \; D = \sqrt{10}\,.$$

This system is not dynamic (fixed **A**, **B**, **C**, **D**).

Example (De Leon): Suppose we wish to describe the trajectory of a projectile shot with an initial velocity of 1m/s at an angle of $\pi / 4$ from the horizon.

Figure

Using basic Newtonian laws of motion (discretized) we have functions for distance (range), $d(n)$ and velocity, $v(n)$:

$$\begin{aligned} d(n) &= d(n-1) + v(n-1)T + \frac{1}{2}a(n-1)T^2 \\ v(n) &= v(n-1) + a(n-1)T \end{aligned}$$

where $a(n)$ is the acceleration and T is the measurement sample period. We can construct two, three-state vectors for each direction (x- and y-):

$$\mathbf{x}_x(n) = \begin{bmatrix} d_x(n) \\ v_x(n) \\ a_x(n) \end{bmatrix}, \mathbf{x}_y(n) = \begin{bmatrix} d_y(n) \\ v_y(n) \\ a_y(n) \end{bmatrix}.$$

In this case the state update equation would be

$$\mathbf{x}(n) = \mathbf{A}(n-1)\mathbf{x}(n-1)$$

where

$$\mathbf{A}(n) = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

initialized with

$$\mathbf{x}_x(0) = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}, \mathbf{x}_y(0) = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ -9.8 \end{bmatrix}$$

We can track the projectile (or determine it trajectory) with the output equation

$$\mathbf{y}(n) = \mathbf{C}(n)\mathbf{x}(n)$$

where $\mathbf{C}(n) = [1 \quad 0 \quad 0]$ (masks off velocity and acceleration in state vector). The absence of $\mathbf{B}(n)$ implies a perfect model of the system while the absence of $\mathbf{D}(n)$ implies we can make perfect observations (measurements). This system is also not dynamic.