Convergence in the Mean Square

Figure 13.2a

For convergence in the mean square we need to show

$$\lim_{n\to\infty} \vartheta(\nu) = \lambda\iota\mu_{\nu\to\infty} E\left[\left|\xi(\nu)\right|^2\right]$$
$$= \chi o\nu\sigma\tau\alpha\nu\tau$$

where

$$\xi(n) = d(n) - \hat{\mathbf{w}}(n-1)\mathbf{u}(n)$$

is the a priori estimation error. We choose $\xi(n)$ instead of the conventional error $e(n) = \delta(\nu) - \hat{\boldsymbol{\omega}}(\nu)\boldsymbol{\upsilon}(\nu)$ due to the design of the RLS update algorithm and its mean-squared value (called MSE here) for comparisons against other algorithms. Substitution of our model for the desired signal yields

$$\xi(n) = [e_o(n) + \mathbf{w}_o^H\mathbf{u}(n)] - \hat{\mathbf{w}}^H(n-1)\mathbf{u}(n)$$
$$= e_o(n) - [\hat{\mathbf{w}}(n-1) - \mathbf{w}_o]^H\,\mathbf{u}(n) \qquad .$$
$$= e_o(n) - \varepsilon^H(n-1)\mathbf{u}(n)$$

Calculation of the MSE yields

$$J'(n) = E[|e_o(n)|^2] - E[\varepsilon^H(n-1)\mathbf{u}(n)e_o^*(n)] - E[e_o(n)\mathbf{u}^H(n)\varepsilon(n-1)] + E[\mathbf{u}^H(n)\varepsilon(n-1)\varepsilon^H(n-1)\mathbf{u}(n)]$$
.

1) $E[|e_o(n)|^2] = \sigma^2$ is the measurement noise variance.

2) The misalignment vector $\varepsilon(n-1)$, is independent of both $\mathbf{u}(n)$ and $e_o(n)$ [which depends on $\mathbf{u}(n)$]. Thus

$$E[\varepsilon^H(\nu-1)\boldsymbol{\upsilon}(\nu)\varepsilon_o^*(\nu)] = E[\varepsilon^H(\nu-1)]E[\boldsymbol{\upsilon}(\nu)\varepsilon_o^*(\nu)]$$

In our model for $d(n)$, $\mathbf{w}_o$ (LS optimal filter) is selected so that the elements of $\mathbf{u}(n)$ are orthogonal to $e_o(n)$. Thus

$$E[\varepsilon^H(\nu-1)]E[\boldsymbol{\upsilon}(\nu)\varepsilon_o^*(\nu)] = E[\varepsilon^H(\nu-1)]\cdot\mathbf{0}$$
$$= 0$$

Same holds true for the complex conjugate case

$$E[e_o(n)\boldsymbol{\upsilon}^H(\nu)\varepsilon(\nu-1)] = 0 \ .$$

3) Based on the independence of the misalignment vector

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and the input $\mathbf{u}(n)$ we have

$$
\begin{aligned}
E[\mathbf{u}^H(n)\varepsilon(n-1)\varepsilon^H(n-1)\mathbf{u}(n)] &= E\{tr[\mathbf{u}^H(n)\varepsilon(n-1)\varepsilon^H(n-1)\mathbf{u}(n)]\} \\
&= E\{tr[\mathbf{u}(n)\mathbf{u}^H(n)\varepsilon(n-1)\varepsilon^H(n-1)]\} \\
&= tr\{E[\mathbf{u}(n)\mathbf{u}^H(n)\varepsilon(n-1)\varepsilon^H(n-1)]\} \\
&= tr\{E[\mathbf{u}(n)\mathbf{u}^H(n)]E[\varepsilon(n-1)\varepsilon^H(n-1)]\} \\
&= tr[\mathbf{PK}(n-1)]
\end{aligned}
$$

Substituting 1) - 3) into our previous MSE calculation we have

$$J'(n) = \sigma^2 + tr[\mathbf{RK}(n-1)] .$$

From our earlier work on the misalignment variance [MSE of $\hat{\mathbf{w}}(n)$] we had

$$\mathbf{K}(n) = \frac{\sigma^2}{n-M-1}\mathbf{P}^{-1}, \ n > M+1 .$$

Assuming $\mathbf{u}(n)$ is white ($\mathbf{R} = \mathbf{I}$) we have

$$J'(n) = \sigma^2 + \frac{M\sigma^2}{n-M-2}, \ n > M+2$$

and

$$\boxed{\lim_{n\to\infty} J'(n) = \sigma^2}.$$

Observations under our assumptions (infinite memory, $\lambda = 1$; stationary environment; $e_o(n)$ small compared to $d(n)$, i.e. high SNR)

1) Convergence in about $2M$ iterations (order of magnitude faster than LMS).
2) Convergence in the mean square is independent of the eigenvalues of $\mathbf{R}$.
3) Theoretically $J'(n)$ approaches variance of the measurement error σ^2. This suggests zero excess MSE or zero misadjustment.

It can be shown that for $\lambda < 1$

$$
\begin{aligned}
J'_{ex} &\approx M\left(\frac{1-\lambda}{1+\lambda}\right)J'_{\min} \\
&= M\left(\frac{1-\lambda}{1+\lambda}\right)\sigma^2
\end{aligned}
$$

thus

$$
\begin{aligned}
\text{Misadjustment} &= \frac{J'_{ex}}{J'_{\min}} \\
&= M\left(\frac{1-\lambda}{1+\lambda}\right).
\end{aligned}
$$

Computer Experiment on Adaptive Equalization

Figure 13.5

Assumptions

Input sequence $x(n) \in \{+1, -1\}$ (Bernoulli).

Channel modeled by

$$h(n) = \begin{cases} \frac{1}{2}\left[1 + \cos\left(\frac{2\pi}{W}(n-2)\right)\right], & n = 1,2,3 \\ 0, & o.w. \end{cases}$$

where *W* controls amplitude distortion produced by channel (and effects the eigenvalue spread of correlation matrix).

Channel noise is white, zero mean, and variance of σ_v^2 .

We choose the length of the equalizer equal to 2*N* + 1 (for a symmetric equalizer) where *N* is the length of the channel (*N* = 5 in this case). Therefore *M* = 11.

Since equalizer (length 11) and optimum filter (length 5) are symmetric they have linear phase. Thus the associated delay in samples for the equalizer and optimum filter is 5 and 2 samples respectively. We therefore set the delay = 7 samples.

RLS with $\lambda = 1$.

Example 1

We assume a channel with 30dB SNR ($\sigma_v^2 = 0.001$).

- Convergence in about 20 samples (~2*M*) and is much faster than LMS
- Relatively insensitive to eigenvalue spread as compared to LMS
- Virtually zero misadjustment.